Lecture notes – The Canonical Portfolio problem

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1 Formulating the portfolio problem

Consider the asset allocation problem of an investor. The investor can invest in a risk free asset, and invest in risky assets. Currently the investor has wealth W_0.

We will consider several variations of the problem.

1.1 One risky, One risk free asset

Suppose the risky asset has return $\tilde{r}$. The risk free asset has return r_f. For convenience define $\tilde{R} = (1 + \tilde{r})$ and $R_f = (1 + r_f)$.

Let ω be the fraction of wealth invested in the risky asset.

The end of period wealth can be calculated using any of the following equivalent formulations

$$\begin{aligned}
\tilde{W} &= W_0(1 + r_f)(1 - \omega) + W_0\omega(1 + \tilde{r}) \\
&= W_0 R_f(1 - \omega) + W_0\omega\tilde{R} \\
&= W_0(R_f + \omega(\tilde{r} - r_f)) \\
&= W_0(R_f + \omega(\tilde{R} - R_f))
\end{aligned}$$

The individual's choice problem

$$\max_{\{\omega\}} E[u(\tilde{W})]$$

An alternative formulation of the same problem is to define $a = W_0\omega$ to be the *amount* invested in the risky asset. Then we find

$$\tilde{W} = (W_0 - a)R_f + a\tilde{R} = W_0 R_f + a(\tilde{R} - R_f)$$

and the choice problem

$$\max_{\{a\}} E[u(\tilde{W})]$$

1.2 One risk free, several risky assets

Suppose the risky asset i has return $\tilde{r}_i$. There are n risky assets. The risk free asset has return r_f. For convenience, define $\tilde{R}_i = (1 + r_i)$ and $R_f = (1 + r_f)$, and let ω_j be the fraction of wealth invested in the risky asset j.

The end of period wealth can be calculated as

$$\tilde{W} = W_0(1 + r_f)\left(1 - \sum_j \omega_j\right) + W_0 \sum_j \omega_j (1 + \tilde{r}_j) = W_0\left(R_f\left(1 - \sum_j \omega_j\right) + \sum_j \omega_j \tilde{R}_j\right)$$

We can also write this problem in vector form. If we define

$$\tilde{\mathbf{R}} = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Then we can write

$$\tilde{W} = W_0 \left((1 - \mathbf{w}'\mathbf{1}) R_f + \mathbf{w}'\tilde{\mathbf{R}} \right)$$

1.3 General formulation

Will also want to use a quite general formulation, with no special role for a risk free asset.

Suppose we have m assets.

Define

p_j as the price for asset j,

$\tilde{x}_j$ as the payoff per unit of asset j, and

n_j as the number of units of asset j hold in a portfolio.

Let $\mathbf{n} = (n_1, n_2, \cdots, n_m)'$, $\mathbf{p} = (p_1, p_2, \cdots, p_m)'$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \cdots, x_m)'$.

Let W_0 be the initial wealth of the agent.

The canonical portfolio problem can be written as

$$\max_{\mathbf{n}} E[u(\tilde{W})]$$

subject to

$$\tilde{W} = \mathbf{n}'\tilde{\mathbf{x}}$$

$$\mathbf{n}'\mathbf{p} \leq W_0$$

We can also write this in returns form. Let

$$\tilde{R}_j = \frac{\tilde{x}_j}{p_j}$$

$$\omega_j = \frac{n_j P_j}{W_0}$$

and let $\mathbf{R} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_m)'$ and $\mathbf{w} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m)'$. Then the canonical portfolio problem can be written as

$$\max_{\mathbf{n}} E[u(\tilde{W})]$$

subject to

$$\tilde{W} = W_0 \mathbf{w}'\mathbf{R}$$

$$\mathbf{w}'\mathbf{1} \leq 1$$

2 When would one choose a risky asset?

Consider the situation with one risky and one risk free asset.

Proposition 1 *In the portfolio problem with one risky and one risk free asset, if the expected return on the risky asset equals the risk free asset($E[\tilde{r}_j] = r_f$), it is an optimal portfolio to invest all in the risk free asset.*

Proposition 2 *In the portfolio problem with one risky and one risk free asset, if the expected return on the risky asset is greater than the risk free asset ($E[\tilde{r}] > r_f$), it is not an optimal portfolio to invest all in the risk free asset.*

3 The relationship between asset choice and risk aversion

Again consider the problem with one risky and one risk free asset.

Proposition 3 *The more risk averse an investor, the higher need* $E[\tilde{r}] - r_f$ *be to induce the investor to hold all his wealth in the risky asset.*

Find by how much need $E[\tilde{r}] > r_f$ to induce a risk averse investor to hold all in the risky portfolio.

4 The relationship between asset choice and initial wealth

4.1 Does wealthier individuals invest more in the risky asset?

Proposition 4 *Suppose the investor is (strictly) risk averse, but displays* decreasing *absolute risk aversion. Then the* amount *a invested in the risky asset increases as the initial wealth* W_0 *increases.*

$$(R_A(z) < 0 \; \forall \; z \quad \Rightarrow \quad \frac{da}{dW_0} > 0)$$

Proposition 5 *Suppose the investor is (strictly) risk averse, but displays* decreasing *relative risk aversion. Then the* fraction ω *invested in the risky asset increases as the initial wealth* W_0 *increases.*

$$(R_A(z) < 0 \; \forall \; z \quad \Rightarrow \quad \frac{d\omega}{dW_0} > 0)$$

4.2 When is asset choice independent of initial wealth?

An interesting issue arises when looking at multiperiod decisions. Are there conditions when the decision problem can be made at the beginning, independent of the outcomes in later periods? This means we are interested in cases where asset allocation decisions are independent of initial wealth. (The myopia problem). For the myopia issue we want to look at constant *proportions* of the risky asset.

Proposition 6 *We consider the portfolio problem with one risky and one risk free asset. Suppose utility is logarithmic* $(U(\tilde{W}) = \ln(\tilde{W}))$*. Then the proportion invested in the risky asset is constant, independent of the initial wealth* (W_0)*.*

But it may also be of interest to ask when the *amount* invested in a risky asset is independent of initial wealth.

Proposition 7 *We consider the portfolio problem with one risky and one risk free asset. Suppose utility is negative exponential* $(U(\tilde{W}) = -e^{-\tilde{W}})$*. Then the amount invested in the risky asset is independent of the initial wealth* (W_0)*.*

5 References and further reading

Huang and Litzenberger (1988) is a good textbook discussion of much of this material.

See Mossin (1968) for a classical discussion of the myopia problem.

References

Chi-fu Huang and Robert H Litzenberger. *Foundations for financial economics*. North–Holland, 1988.

Jan Mossin. Optimal multiperiod portfolio problems. *Journal of Business*, 41(2):215–229, 1968.