Lecture notes – Mean Variance Analysis

December 21, 2023

1 The representation of preferences using mean-variance representations

In a number of applications in finance, preferences are represented with a special kind of utility function, where the decision maker only cares about the mean and variance of the portfolio return distribution.

$$u(p) = u(E[r_p], \sigma(r_p))$$

Decision makers are assumed to prefer higher expected return ($\partial u/\partial E[r_p] > 0$) and to dislike increased variance ($\partial u/\partial \mathrm{var}(r_p) < 0$).

Definition 1 (Mean variance utility representation) *We say that preferences over portfolios of assets can be represented with a* mean variance utility representation $u(p)$*, where* p *is the portfolio,*

$$u(p) = u(E[r_p], \sigma(r_p))$$

in such a manner that $u(\cdot,\cdot)$ *is an increasing function of* $E[r_p]$ *and a decreasing function of* $\sigma(r_p)$*, where* r_p *is the portfolio return.*

How can such a utility representation be justified?

Proposition 1 *The choices of a risk–averse, nonsatiated individual with quadratic preferences satisfy a mean variance utility representation.*

Proposition 2 *Suppose returns are normally distributed. Then preferences can be represented with a mean variance utility representation.*

For general utility functions a mean-variance representation can be motivated by a Taylor expansion around expected future wealth:

$$u(\tilde{W}) = u(E[\tilde{W}] + u'(E[\tilde{W}])(\tilde{W} - E[\tilde{W}]) + \frac{1}{2}u''(E[\tilde{W}])(\tilde{W} - E[\tilde{W}])^2 + h.o.t.$$

Under a number of assumptions about the applicability of this expansion, the expected utility function can be expressed as

$$E[u(W)] \approx u(E[W]) + \frac{1}{2}u''(E[W])\sigma^2(W)$$

Since $u' > 0$ and $u'' < 0$ this means that for the utility functions for which this is a reasonable approximation utility is increasing in expected wealth and decreasing in variance of expected wealth, which also implies that it has the same signs for returns, since $\tilde{W} = W_0(1 + \tilde{r})$.

2 The mathematics of the Mean Variance Frontier.

This part will cover the basic results about the mean variance frontier.

We assume that investors preferences over portfolios p satisfy a mean variance utility representation, $u(p) = u(E[r_p], \sigma(r_p))$, with utility increasing in expected return ($\partial u/\partial E[r_p] > 0$) and decreasing in variance ($\partial u/\partial \text{var}(r_p) < 0$).

In this part we consider the representation of the *portfolio opportunity set* of such decision makers. There are a number of useful properties of this opportunity set which follows purely from the mathematical formulation of the optimization problem. It is these properties we focus on here.

3 Setup.

Assumption 1 (Securities) *There exists $n \geq 2$ risky securities, with expected returns* $\mathbf{e}$

$$\mathbf{e} = \begin{bmatrix} E[r_1] \\ E[r_2] \\ \vdots \\ E[r_n] \end{bmatrix}$$

and covariance matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \sigma(r_1, r_1) & \sigma(r_1, r_2) & \dots & \\ \sigma(r_2, r_1) & \sigma(r_2, r_2) & \dots & \\ \vdots & & & \\ \sigma(r_n, r_1) & & \dots & \sigma(r_n, r_n) \end{bmatrix}$$

Assumption 2 *The covariance matrix* $\mathbf{V}$ *is invertible.*

Definition 2 (Portfolio) *A portfolio p is defined by a set of weights* $\mathbf{w}$ *invested in the risky assets.*

$$\mathbf{w} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

The expected return on a portfolio is calculated as

$$E[r_p] = \mathbf{w}'\mathbf{e}$$

and the variance of the portfolio is

$$E[r_p] = \mathbf{w}'\mathbf{V}\mathbf{w}$$

4 The minimum variance frontier

Definition 3 (Frontier Portfolio) *A portfolio is a* frontier *portfolio if it minimizes the variance for a given expected return.*

That is, a frontier portfolio p solves

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

s.t.

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

Definition 4 (Minimum Variance Frontier) *The set of all frontier portfolios is called the* minimum variance frontier.

5 Calculation of frontier portfolios.

Proposition 3 *If the matrix* $\mathbf{V}$ *is full rank, and there are no restrictions on shortsales, the weights* $\mathbf{w}_p$ *for a frontier portfolio* p *with mean* $E[\tilde{r}_p]$ *can be found as*

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}E[r_p]$$

where

$$\mathbf{g} = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1}$$

$$\mathbf{h} = \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}$$

$$A = \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

$$B = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

$$C = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$\mathbf{A} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}$$

$$D = BC - A^2 = |\mathbf{A}|$$

Remark 1 *The portfolio defined by weights* $\mathbf{g}$ *is a portfolio with expected return 0. The portfolio defined by weights* $(\mathbf{g}+\mathbf{h})$ *is a portfolio with expected return 1. Note also the useful property that* $\mathbf{g}\mathbf{1}' = 1$*, and* $\mathbf{h}\mathbf{1}' = 0$*.*

Proposition 4 *The portfolios* $\mathbf{g}$ *and* $(\mathbf{g}+\mathbf{h})$ *generate the entire minimum variance frontier.*

Proposition 5 *Any two portfolios on the minimum variance frontier with distinct means are sufficient to generate the minimum variance frontier.*

Proposition 6 *The covariance between any two frontier portfolios* p *and* q *can be found as*

$$cov(r_p, r_q) = \frac{C}{D}\left(E[r_p] - \frac{A}{C}\right)\left(E[r_q] - \frac{A}{C}\right) + \frac{1}{C}$$

By setting $q = p$, this result also let us to describe the variance of any frontier portfolio.

Proposition 7 *The variance of any frontier portfolio* p *can be written as*

$$var(r_p) = cov(r_p, r_p) = \frac{C}{D}\left(E[r_p] - \frac{A}{C}\right)^2 + \frac{1}{C}$$

6 The global minimum variance portfolio

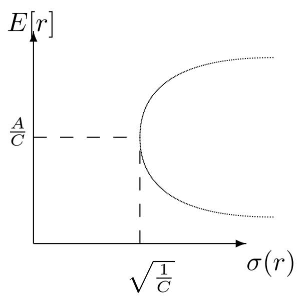
Definition 5 (Global Minimum Variance Portfolio) *The portfolio that minimizes variance regardless of expected return is called the* global minimum variance portfolio*.*

Let *mvp* be the global minimum variance portfolio.

Proposition 8 (Global Minimum Variance Portfolio) *The minimum variance portfolio mvp has expected return* $E[r_{mvp}] = \frac{A}{C}$ *and variance* $var(r_{mvp}) = \frac{1}{C}$.

Proposition 9 (Global minimum variance portfolio) *The global minimum variance portfolio has weights*

$$\mathbf{w}'_{mvp} = \left(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}'\mathbf{V}^{-1} = \frac{1}{C}\mathbf{1}'\mathbf{V}^{-1}$$

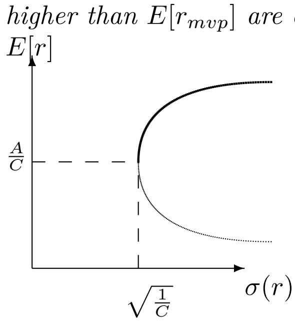


Proposition 10 *For any portfolio* $p \neq mvp$,

$$cov(r_p, r_{mvp}) = var(r_{mvp})$$

7 Efficient portfolios.

Definition 6 (Efficient Portfolio) *Portfolios on the minimum variance frontier with expected returns higher than* $E[r_{mvp}]$ *are called* efficient



Proposition 11 *All portfolios on the positively–sloped segment of the efficient set are positively correlated.*

Proposition 12 *The proportion of an efficient portfolio invested in a given individual asset changes monotonically along the efficient frontier.*

Proposition 13 *Any combination of frontier portfolios is on the frontier.*

Proposition 14 *Any combination of efficient frontier portfolios is efficient.*

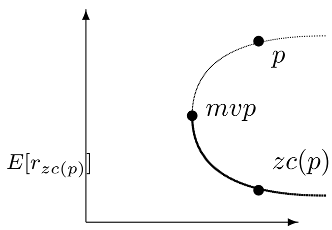
8 The zero beta portfolio

Proposition 15 *For any portfolio p on the frontier, there is a frontier portfolio $zc(p)$ satisfying*

$$cov(r_{zc(p)}, r_p) = 0.$$

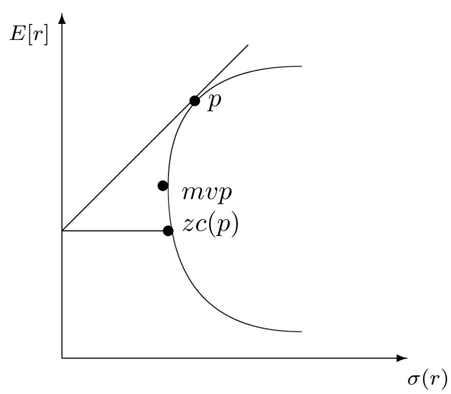
This portfolio is called the zero beta portfolio relative to p. The zero beta portfolio $zb(p)$ has return

$$E[r_{zc(p)}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[r_p] - \frac{A}{C}}$$

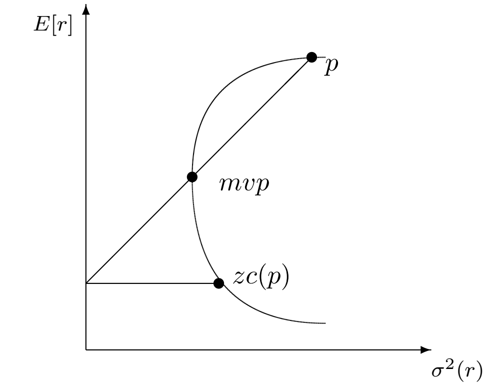


Corrolary 1 *p is a portfolio on the mv frontier. If p is efficient $zc(p)$ is inefficient. If p is inefficient $zc(p)$ is efficient.*

Proposition 16 *In $E[r]-\sigma(r)$ space, $E[r_{zc(p)}]$ is found on intercept of the tangency line at p on the efficient frontier.*



Proposition 17 *In $E[r] - \sigma^2(r)$ space the line from p to $E[r_{zc(p)}]$ goes through the minimum variance portfolio mvp.*



9 Relationship between any portfolio q and frontier portfolios

Proposition 18 *For any portfolio q, if p is a frontier portfolio, we can express $E[r_q]$ as a linear function of $E[r_p]$:*

$$E[r_q] = E[r_{zc(p)}] + \beta_{qp}(E[r_p] - E[r_{zc(p)}])$$

where

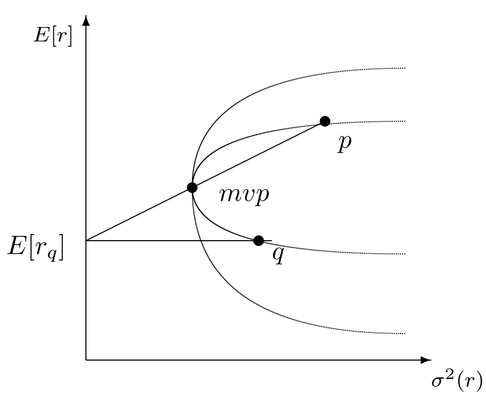
$$\beta_{qp} = \frac{cov(r_q, r_p)}{var(r_q)}$$

The above proposition actually is an if and only if statement, Roll (1977) proved the following more general proposition.

Proposition 19 *The covariance vector of individual assets with any portfolio can be expressed as an exact linear function of the individual mean returns vector if and only if the portfolio is efficient.*

Remark 2 *This is the basis for Roll's claim that the only possible test of the CAPM is that the market portfolio is mean variance efficient.*

Proposition 20 *Consider any portfolio p* not *on the portfolio frontier. The intercept on the expected rate of return axis of a line from p though mvp is equal to the expected return on a portfolio q, which has zero covariance with p and has the minimum variance among all the zero covariance portfolios with p.*



10 Allowing for a riskless asset.

Suppose have N risky assets with weights $\mathbf{w}$ and one riskless assets with return r_f.

Intuitively, the return on a portfolio with a mix of risky and risky assets can be written as

$$E[r_p] = \text{weight in risky} \times \text{ return risky } + \text{weight riskless } \times r_f$$

which in vector form is:

$$E[r_p] = \mathbf{w}'\mathbf{e} + (1 - \mathbf{w}'\mathbf{1})r_f$$

Proposition 21 *An efficient portfolio in the presence of a riskless asset has the weights*

$$\mathbf{w}_p = \mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)\frac{E[r_p] - r_f}{H}$$

where

$$H = (\mathbf{e} - \mathbf{1}r_f)'\mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)$$

Remark 3 *Note that $H > 0$ since it is a quadratic form.*

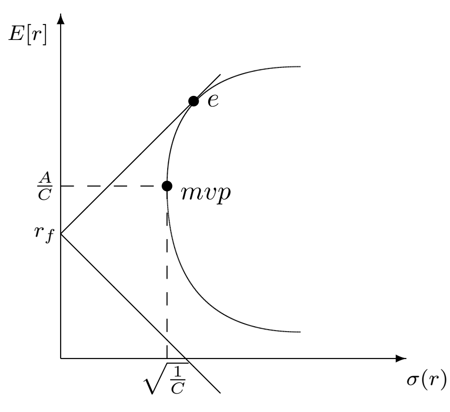
Proposition 22 *The variance of the efficient portfolio is*

$$\sigma^2(r_p) = \frac{(E[r_p] - r_f)^2}{H}$$

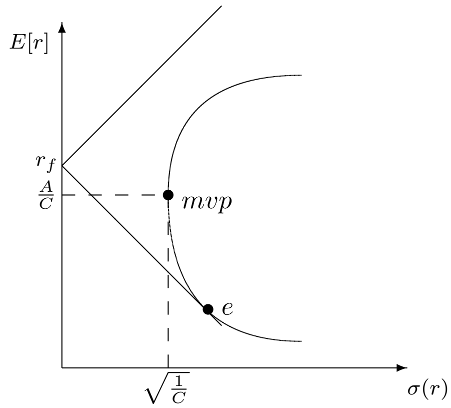
Remark 4 *Note that standard deviationn is a linear function of $E[r_p]$. The efficient set is a line in mean-standard deviation space.*

11 Efficient sets with risk free assets.

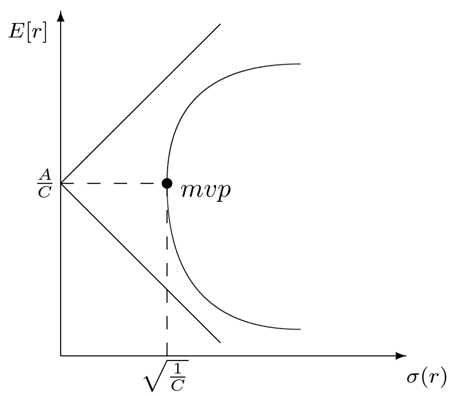
Proposition 23 *Suppose $r_f < \frac{A}{C}$. Then the efficient set is the line from $(0, r_f)$ through tangency on the efficient set of* risky *assets.*



Proposition 24 *Suppose $r_f > \frac{A}{C}$. Then the efficient set is the two half-lines starting from $(0, r_f)$.*



Proposition 25 *If $r_f = \frac{A}{C}$, the weight in the risk free asset is one. The risky portfolio is an zero investment portfolio. The efficient set consists of two asymptotes toward the efficient set of risky assets.*



12 Relation between any portfolio and frontier portfolios with risk free asset.

Proposition 26 *If there exist a risk free rate r_f, for any portfolio q and frontier portfolio p, the expected return $E[r_q]$ can be written as a linear function of $E[r_p]$:*

$$E[r_q] = r_f + \beta_{qp}(E[r_p] - r_f)$$

13 The Sharpe Ratio

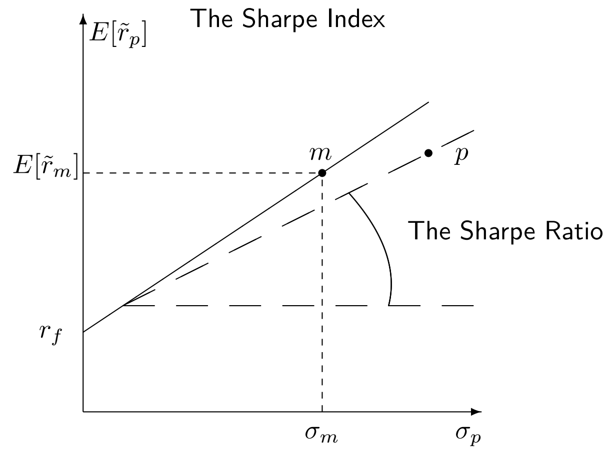
Definition 7 (The Sharpe Ratio) *The Sharpe ratio of a given portfolio p is defined as*

$$S_p = \frac{E[r_p] - r_f}{\sigma(r_p)}$$

Proposition 27 *The Sharpe ratio S_p of a portfolio p is the slope of the line in mean-standard deviations space from the risk free rate through p.*

Proposition 28 *The tangency portfolio has the maximal Sharpe Ratio on the efficient frontier.*

Example



Proposition 29 *If p is an efficient portfolio and q is any portfolio then*

$$\rho(r_p, r_q) = \frac{S_p}{S_q}$$

14 Short-sale constraints.

So far the analysis has put no restrictions on the set of weights w_p that defines the minimum variance frontier. For practical applications, existence of negative weights is problematic, since this involves selling securities short.

This has led to the investigation of *restricted* mean variance frontiers, where the weights are constrained to be non-negative.

Definition 8 *A short sale resctricted minimum variance portfolio p solves*

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

s.t.

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

$$\mathbf{w}' \geq \mathbf{0}$$

Such short sale resctricted minimum variance portfolio portfolios are much harder to deal with analytically, since they do not admit a general solution, one rather has to investigate the Kuhn-Tucker conditions for corner solutions etc.

14.1 Conditions under which all portfolio weights are optimally positive

To avoid having to deal with shortsale constraints an obvious question to ask is whether there are conditions on $\mathbf{e}$ and $\mathbf{V}$ which guarantees that optimal weights will be positive. Unfortunately no simple answer here. See some comments in Roll (1977) and Green (1986).

15 Further issues.

Backing out expected returns from weights and $\mathbf{V}$.

16 The mean variance consequences of the tracking error criterion

While we are on the topic of mean variance mathematics one applied topic may serve to show the usefulness of knowledge of this particular mathematical representation. Consider the case of *tracking error*

16.1 Mutual funds

Mutual fund: “a low cost way for the investor in the street to own a portfolio of securities”.

Two main types:

- Index funds: A well diversified portfolio that tries to match a broad equity index, such as S&P 500.
- Active funds: Funds that tries to get a superior return by active managment. Activity can be such as
  - stock picking (superior information about indivdual stocks)
  - timing, moving between asset classes. (better at timing, when is stocks doing better than bonds, or cash?

Elements that enter into the choice of mutual funds for individual investors

- Direct costs for holding the fund: Typical: Annual percentage of net asset value, and for active funds: performance fees.
- Tradings costs incurred by the fund in its trading - ”hidden” to the mutual fund investors. Examples: Soft dollar commisions, directed brokerage.
- Question of active management: Does active portfolio managers really ”add value”?

In choosing a mutual fund, important decision problem:

- Asset class mix.
- Degree of activity.

However, once this decision made, want to evaluate the degree to which a mutual fund manager achieve the goal.

How to operationalize this.

Well, here is one way:

Let the mutual fund investor give the manager a *benchmark*. This benchmark is the return of an (ex ante) specified index with a given mix of asset classes: E.g. the Norwegian Pension (Oil) fund: 60/40 bonds/equity, with a specified country mix.

If all the investor wants is a passive index matching, demand that the mutual fund returns match this index.

However, an ”active” manager will have to better than that, the active manager should provide something more than the index.

Thus, the benchmark serves as a yeardstick which the manager should beat. But the manager should not beat the benchmark at any cost. This means that we should worry about the difference between the benchmark and the actual portfolio, the *tracking error*.

To operationalize such a multidimensional criterion the typical formulation is to tell the manager to minimize the variance of the tracking error subject to achieving a given expected return above the benchmark (tracking error).

This type of criterion has emerged as an important feature of the mutal fund industry.

What we will do here is to use the Mean Variance framework already seen to show that this tracking error criterion has some drawbacks.

In a mean variance framework it is sensitive to the efficiency of the chosen benchmark and is as such open to the Roll (1977) critique. It is only natural that Roll used the same mathemetics as in his ’77 CAPM critique to make a similar point about the (non) efficiency of tracking error portfolios.

16.2 Analyzing the tracking error criterion in a mean variance framework

Definition 9 (Tracking error) *A tracking error is defined relative to a benchmark portfolio. The tracking error of any portfolio relative to the benchmark is the difference in return between the portfolio and the benchmark.*

Definition 10 (TEV criterion) *The Tracking Error Variance criterion for portfolios is to minimize the variance of the tracking error subject to achieving a given desired expected tracking error.*

To formulate the problem, define

- $\mathbf{e}$ as the vector of expected returns,
- $\mathbf{V}$ as the covariance matrix and
- $\mathbf{w}$ as the vector of weights defining a portfolio.

Any portfolio p is fully defined by its associated weights $\mathbf{w}_p$. The portfolio has expected return $E[r_p] = \mathbf{w}_p'\mathbf{e}$ and variance $\sigma^2(r_p) = \mathbf{w}_p'\mathbf{V}\mathbf{w}_p$.

A portfolio of particular interest is the *benchmark* portfolio b, specified by its associated weights $\mathbf{w}_b$.

We consider the properties of the portfolio p chosen by the asset manager. Again it is fully specified by its weights $\mathbf{w}_p$. However, it can alternatively be described by a *tracking error portfolio* $\mathbf{x}_p$ which specifies the changes made from the benchmark portfolio to achieve p, or the *difference* between the benchmark b and the portfolio p:

$$\mathbf{w}_p = \mathbf{w}_b + \mathbf{x}_p,$$

or

$$\mathbf{x}_p = \mathbf{w}_p - \mathbf{w}_b.$$

The *tracking error* is then

$$r_p - r_b = \mathbf{e}'(\mathbf{w}_p - \mathbf{w}_b) = \mathbf{e}'\mathbf{x}_p$$

and the *tracking error variance* (TEV) is

$$\sigma^2(r_p - r_b) = (\mathbf{w}_p - \mathbf{w}_b)'\mathbf{V}(\mathbf{w}_p - \mathbf{w}_b) = \mathbf{x}_p'\mathbf{V}\mathbf{x}_p$$

The exact formulation of the optimization problem, given a desired tracking error G, is:

$$\mathbf{x}_p = \arg\min_{\mathbf{x}} \ \frac{1}{2}\mathbf{x}'\mathbf{V}\mathbf{x}$$

subject to

$$\mathbf{x}'\mathbf{e} = G$$

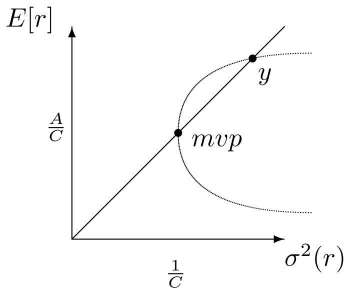
$$\mathbf{x}'\mathbf{1} = 0$$

Proposition 30 *The optimal weights for the tracking error portfolio x_p with the desired tracking error G has the form*

$$x_p = \frac{G}{E[r_y] - E[r_{mvp}]}(\mathbf{w}_y - \mathbf{w}_{mvp})$$

where mvp is the minimum variance portfolio and y is another frontier portfolio with expected return $E[r_y] = \frac{B}{A}$.

Remark 5 *The portfolio y lies on a line from the origin through mvp in mean-variance space.*



Corrolary 2 *The optimal x_p relative to a benchmark b is independent of the benchmark, it only depends on the desired tracking error G.*

I.e. No matter what benchmark, for a given tracking error, everybody will make the same set of trades.

Proposition 31 *The portfolio x_p has variance*

$$\sigma^2(r_{x_p}) = K\left(\sigma^2(r_y) - \sigma^2(r_{mvp})\right)$$

where $K = \frac{G}{E[r_y]-E[r_{mvp}]}$.

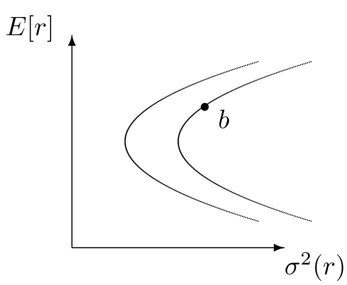
Proposition 32 *The portfolio w_p has variance*

$$\sigma(r_p) = \sigma^2(r_b) + \sigma^2(r_{x_p}) + 2K\sigma^2\left(r_{mvp}\right)\left(\frac{E[r_b]}{E[r_{mvp}]} - 1\right)$$

16.3 Implications of TEV optimization in M-V space

Definition 11 (TEV frontier) *The set of TEV portfolios optimal relative to a given benchmark, with varying desired tracking error G is called the TEV frontier relative to the benchmark.*

Proposition 33 *The distance between the TEV frontier for a given benchmark and the global mv frontier is constant.*



Corrolary 3 *If a given benchmark portfolio is M-V inefficient, all its associated TEV optimal portfolios will be M-V inefficient.*

16.4 Concluding

Tracking error and its variations seems sensible, and may still be sensible:

Important purpose: Simplifies the monitoring problem of principals, which is an important feature.

16.5 References

Mahoney (2004), good on the conflicts on mutual fund managers.

Roll (1992) for the mean variance stuff

17 The relation between complete markets and mean-variance pricing.

Show an negative result, impossible with complete agreement between mean-variance optimization and complete markets.

Due to Dybvig and Ingersoll (1982).

Assumption 3 (Perfect markets) *The markets for all assets are perfect with no taxes or transactions costs. Unlimited borrowing and short selling are permitted with full use of the proceeds. Each asset is infinitely divisible.*

Assumption 4 (Competition) *All investors act as price takers in all markets.*

Assumption 5 (Homogenous expectations) *All investors have identical probability beliefs*

Assumption 6 (State-independent utility) *Investors are risk averse and maximize the expectation of a von Neuman–Morgenstern utility function which depends solely on wealth.*

Assumption 7 (Complete Markets) *Each competitive investor can obtain any pattern of returns through the purchase of marketed assets (subject only to his own budget constraint). If the number of outcome states is finite, markets are complete if the number of marketed assets with linearly independent returns is equal to the number of states.*

Proposition 34 *Given the above assumption, supposing mean-variance pricing holds for all assets, and markets are complete. Then this can not be an equilibrium, since arbitrage opportunities will exist.*

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