Linear Pricing

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In our state price work we have *assumed* the existence of (implicit) state prices, which were either found directly from prices of state contingent claims or implied from unravelling prices of state contingent claims from prices of complex securities.

However, the *existence* of state price can be proven from much less restrictive assumptions.

This is however just a special case of a more general result, namely the linearity of price operators.

Consider a market that pay off in different states of nature. (These states need not be full-fledged Arrow-Debreu states, only outcomes of random processes.

Definitions

Let $X_{i\omega}$ be the payoff for security i in state ω. There are I assets and W states. Let w_i be the number of asset i held in a portfolio.

$$\mathbf{X} = \begin{bmatrix} X_{11} & .. & X_{1I} \\ . & & \\ X_{W1} & .. & X_{WI} \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} W_1 \\ . \\ W_I \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ . \\ C_I \end{bmatrix} = \begin{bmatrix} X_{11} & .. & X_{1I} \\ . & & \\ X_{W1} & .. & X_{WI} \end{bmatrix} \begin{bmatrix} W_1 \\ . \\ W_I \end{bmatrix}$$

or

$$\mathbf{C} = \mathbf{XW}$$

Let $\mathbf{P}$ be the vector of prices for the I securities.

$$\mathbf{P} = \begin{bmatrix} P_1 \\ . \\ P_I \end{bmatrix}$$

What is an arbitrage opportunity?

A free lunch, i.e, something that pays off a positive amount with positive probability, is guaranteed to never have negative payoffs, i.e. $\mathbf{C} > 0$, or $\mathbf{XW} > 0$ (note that this is the vector definition of strictly positive) and has zero or negative cost today, i.e. $\mathbf{PW} \leq 0$.

An assumption of no arbitrage opportunities is then formalized as

If

$$\mathbf{XW} \leq 0$$

then

$$\mathbf{PW} \geq 0$$

1 The fundamental theorem of finance

Proposition 1 (Fundamental Theorem of Finance) *If the economy does not admit arbitrage opportunities there exists a vector of nonnegative* state prices

$$\phi = (\phi_1, \phi_2, \ldots, \phi_W)$$

such that the price of any existing asset i is given by

$$p_i = \sum_{\omega} \phi_\omega X_{i\omega} = \phi \mathbf{X}_i$$

2 Risk Neutral Valuation

An equivalent way of formulating linear pricing: Price by discounting an "expectation" by the risk free rate.

Proposition 2 (Risk Neutral Valuation) *Suppose an asset is priced as in proposition 1 Then any asset can be alternatively be valued as*

$$P_j = \frac{1}{1+r_f} \sum_{\omega} q_\omega x_{j,\omega} = \frac{1}{1+r_f} E^* [x_j]$$

where q is a vector of "state price probabilities", i.e. the vector q sum to one and all $q_i \geq 0$ and r_f is the risk free rate.

References Varian (1987)

References

Hal Varian. The arbitrage principle in financial economics. *Journal of Economic Perspectives*, 1(2):55–72, 1987.