Lecture notes – Expected Utility and Risk Aversion

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Contents

1 Expected Utility representation and use.

This part of the course will cover much of the same material as what is in the first two chapters of Huang and Litzenberger (1988), but in a somewhat different ordering.

Main topics are

- The representation of choice under uncertainty by an (von Neumann–Morgenstern) *expected utility function*.
- Limitations of expected utility (violations of the independence axiom)
- Measures of Risk Aversion.

- Implications of Risk Aversion.
- Stochastic Dominance.

Good alternative sources for the material are Mas-Colell, Whinston, and Green (1995) and Hirshleifer and Riley (1992)

2 Representing preferences by expected utility

Primitives of the decision problem facing an individual

- A set of states
- A set of acts
- A consequence function giving outcomes as a function of acts and states.
- A probability function for the set of states (objective/subjective)
- A preference scaling function

Preference Relation

Let X be the set of possible consumption plans. A preference relation $\succeq$ is a binary relation on X.

A preference relation is *rational* if it is

1. Complete: $\forall x, y \in X,$ either $x \succeq y$ or $y \succeq x$ or both.
2. Transitive: $\forall x, y, z \in X$, If $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

Risky alternatives

Suppose X is a finite set, $x_i \in X, i = 1, 2, \ldots n$. For each element x_i of X we can assign a probability p_i.

A *lottery* or *consumption plan* P is a vector of probabilities for each outcome. Let $\mathcal{P}$ be the set of all lotteries.

Definition 1 *The preference relation $\succeq$ on the space of lotteries $\mathcal{P}$ is* continuous *if for any $P, P', P'' \in \mathcal{P}$: if$P \succeq P' \succeq P''$ then there are $\alpha, \beta \in (0,1)$ such that $\alpha P + (1-\alpha)P'' \succeq P' \succeq \beta P + (1-\beta)P''$*

Definition 2 *The preference relation $\succeq$ on the space of lotteries $\mathcal{P}$ satisfies the* independence axiom *if $\forall P, P', P'' \in \mathcal{P}$ and $\alpha \in (0,1)$,*

$$P \succeq P' \text{ iff } \alpha P + (1-\alpha)P'' \succeq \alpha P' + (1-\alpha)P''$$

(If we mix two lotteries with a third one, the ordering does not depend on the third one.)

Definition 3 *The utility function U has an* expected utility form *if there is an assignment of numbers $u_1, u_2, \ldots, u_n$ to the n outcomes such that $\forall P \in \mathcal{P}$ we have*

$$U(P) = u_1 p_1 + \ldots + u_n p_n$$

A utility function $U : \mathcal{P} \to \mathbb{R}$ with the expected utility form is called a von Neumann–Morgenstern expected utility function.

Proposition 1 (Linearity of U) *A utility function $U : \mathcal{P} \to \mathbb{R}$ has an expected utility form if and only if it is* linear*, that is, if and only if it satisfies the property that*

$$U\left(\sum_{k=1}^{K} \alpha_k P_k\right) = \sum_{k=1}^{K} \alpha_k U(P_k)$$

for any K lotteries $P_k \in \mathcal{P}$ and probabilities $\alpha_i \in (0,1)$, $\sum \alpha_i = 1$

Proposition 2 *Suppose that $U : \mathcal{P} \to \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succeq$ on $\mathcal{P}$. Then $\bar{U}$ is another v.N-M expected utility function for $\succeq$ if and only if there are scalars $\beta > 0$ and γ such that $\bar{U}(P) = \gamma + \beta U(P)$ for any $P \in \mathcal{P}$.*

Theorem 1 (Expected Utility Theorem) *Suppose that the rational preference relation $\succeq$ on the space of lotteries $\mathcal{P}$ satisfies the continuity and independence axioms. Then $\succeq$ admits a utility representation of the expected utility form. That is, we can assign a number u_i to each outcome x_i such that for any two lotteries P and P', we have*

$$P \succeq P' \text{ iff } \sum_{i=1}^{n} u_i p_i \geq \sum_{i=1}^{n} u_i p_i'$$

Will not prove this theorem. See the discussion in Mas-Colell et al. (1995) or Huang and Litzenberger (1988).

With some technical additions it can be extended to infinite dimensional outcome spaces.

This theorem is extremely important, since it allows us to do all our analysis in terms of properties of the *utility function* $u(x)$, which is very convenient analytically.

3 Problems with expected utility

Main problem: Independence axiom.

See Machina (1987) for some discussion.

3.1 The Allais Paradox.

Exercise 1.

Consider two alternative gambles

Gamble A: 1 million with certainty

Gamble B:

$$\begin{cases} \text{5mill} & w.p & 0.1 \\ \text{1mill} & w.p & 0.89 \\ 0 & w.p & 0.01 \end{cases}$$

Next, consider the two alternative gambles

Gamble C:

$$\begin{cases} \text{1mill} & w.p & 0.11 \\ 0 & w.p & 0.89 \end{cases}$$

Gamble D:

$$\begin{cases} \text{5mill} & w.p & 0.1 \\ 0 & w.p & 0.9 \end{cases}$$

1. Would you say that it is a reasonable preference to prefer A to B? ($A \succeq B$.)
2. Would you also say that it is is a reasonable preference to prefer D to C ($D \succeq C$).
3. Show that an individual who claims both these preferences violates expected utility.

Is this worrying? Depends.

Parallel to optical illusion. Question: Will you want to change your choices if the inconcistency is pointed out to you?

For more on these worrying aspects of expected utility, see Machina (1987) and Machina (1989).

4 Monetary outcomes.

Will for convenience in most of the analysis work with unidimensional outcomes, which for interpretation is easiest thought of as money. Alternatively one can think of consumption, measured by the monetary value of consumption. (Supposing consumption is best choice given constraints.)

5 Some Common Utility Functions

These are some of the common utility functions. a, b, c and d are constants.

Linear

$$u(x) = a + bx$$

Quadratic

$$u(x) = a + bx + cx^2$$

Power Utility

$$u(x) = a(b + cx)^d$$

Exponentional

$$u(x) = ae^{bx}$$

Logarithmic

$$u(x) = a\ln(bx)$$

Will return to these, for example: What are the necessary restrictions on the constants?

6 Multiperiod utility functions.

Extension to multiperiod problems obvious.

Let C_t be the consumption in period t, and let $\mathbf{C} = \{C_1, C_2, \cdots, C_T\}$ be a *consumption plan.* Utility of the consumption plan is

$$U(\mathbf{C}) = U(\{C_1, C_2, \cdots, C_T\})$$

Typical assumptions about this function is *additive separability*

$$U(\mathbf{C}) = \sum_{t=1}^{T} U(C_t, t)$$

If utility is additively separable, time preference can be defined as the rate of substitution required between consumption in two consecutive periods to maintain utility at a given level when the original allocation was equal

$$\rho(C_t, t) = \left.\frac{dC_{t+1}}{dC_t}\right|_{U=\overline{U}, C_t=C_{t+1}=\overline{C}} = \frac{-\frac{\partial U}{\partial C_t}}{-\frac{\partial U}{\partial C_{t+1}}}$$

A further restriction on the utility function is to assume a *constant* time preference. If we do, which is very common in economic problems, we can write utility as

$$U(\mathbf{C}) = \sum_{t=0}^{T} \beta^t U(C_t)$$

where β is the *discount factor* that measures the rate of time preference.

7 Risk Aversion

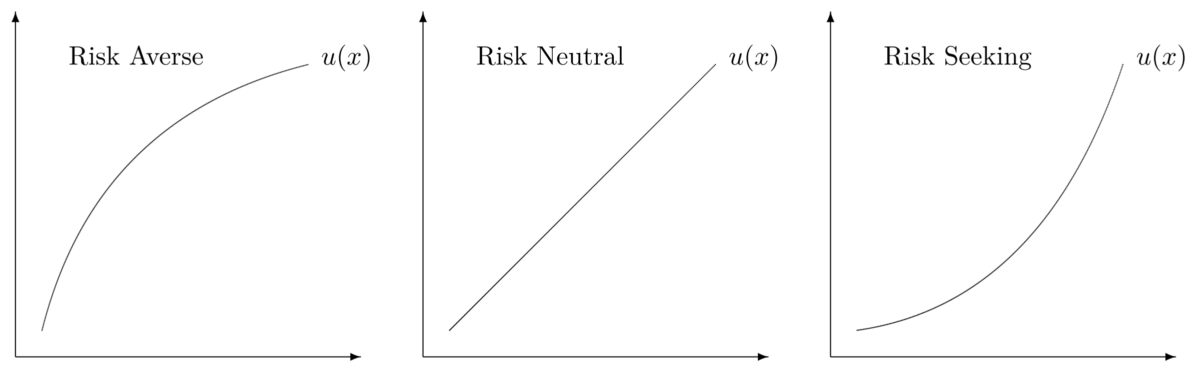
We now turn to risk aversion and its measurement and implications.

For simplicity work with unidimensional set of outcomes, easiest thought of as money.

We assume the existence of a v.N-M expected utility function. $u(x)$, which can be used to rank outcomes.

Definition 4 (Risk Aversion) *A person is risk averse (displays risk aversion) if he strictly prefers a certainty consequence to any risky prospect whose mathematical expectation of consequences equals that certainty. If his preferences go the other way he is a risk–preferrer (displays* risk–preference*). If he is indifferent between the certainty consequence and such a risky prospect he is* risk–neutral *(displays* risk–neutrality*).*

Proposition 3 *The utility function u for a risk averse individual is concave.*



Definition 5 (Certainly Equivalent) *Given a utility function $u(x)$ and a probability distribution F, we define the* certainty equivalent *$CE(F, u)$ as the amount of money for which the individual is indifferent between the gamble F and the certain amount $CE(F, u)$, that is*

$$u(CE(F, u)) = E[u(x)]$$

Proposition 4 *For a risk averse investor $CE(F, u) \leq E[x]$ for all gambles F.*

8 Risk aversion measures.

Definition 6 (Absolute Risk Aversion) *Given a (twice differentiable) utility function $u()$, the* Arrow–Pratt coefficient of absolute risk aversion *at x is defined as*

$$R_A(x) = -\frac{u''(x)}{u'(x)}$$

Definition 7 (Decreasing Absolute Risk Aversion) *The utility function $u(x)$ exhibits* decreasing absolute risk aversion *if $R_A(x)$ is a decreasing function of x.*

Definition 8 (Relative Risk Aversion) *Given a (twice differentiable) utility function $u()$, the* coefficient of relative risk aversion *at x is*

$$R_R(x) = -\frac{xu''(x)}{u'(x)} = xR_A(x)$$

9 Some common utility functions.

9.1 Quadratic utility.

Exercise 2.

Suppose we are using a quadratic utility function

$$u(x) = a + bx + cx^2$$

1. What are the relevant restrictions on the coefficients that follows from nonsatiation and risk aversion?
2. Show that the quadratic utility specification for a risk averse individual implies satiation.

Figure 1 illustrates the problem of satiation:

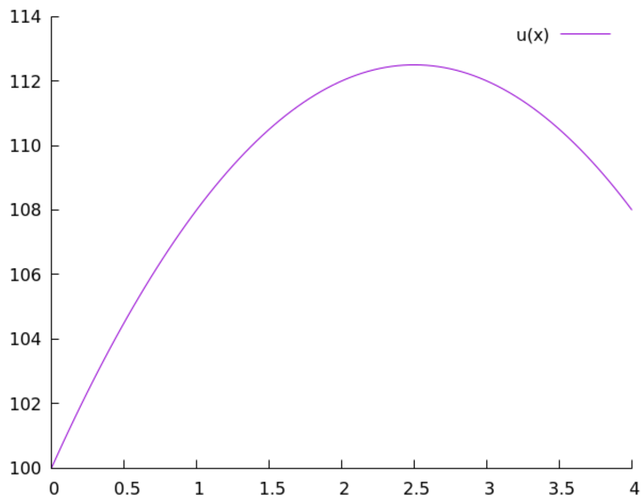


Figure 1: Quadratic Utility Function $u(x) = a + bx + cx^2$, $a = 100$, $b = 10$, $c = -2$

9.2 Logarithmic Utility

Exercise 3.

Suppose we are using a logarithmic utility function

$$u(x) = a + b\ln(cx)$$

1. What are the relevant restrictions on the coefficients that follows from nonsatiation and risk aversion?

Figure 2 illustrates the utility function

9.3 Exponential Utility

Exercise 4.

Suppose utility is exponential

$$u(x) = a + be^{cx}$$

1. What are the relevant restrictions on the coefficients that follows from nonsatiation and risk aversion?

Figure 3 illustrates the utility function

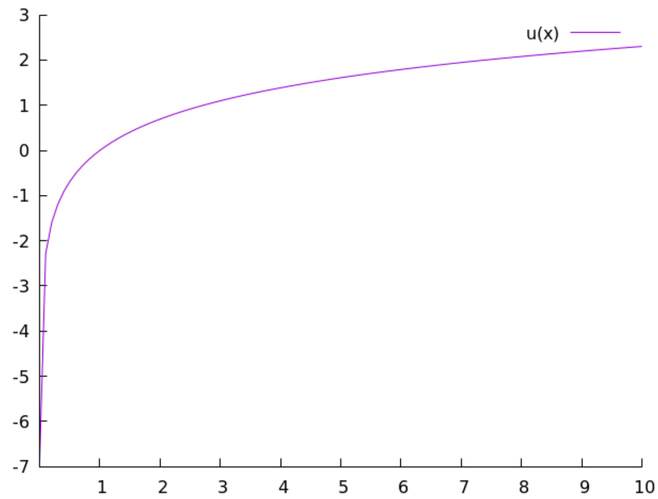


Figure 2: Logarithmic Utility Function $u(x) = a + b\ln(cx)$, $a = 0$, $b = 1$, $c = 1$

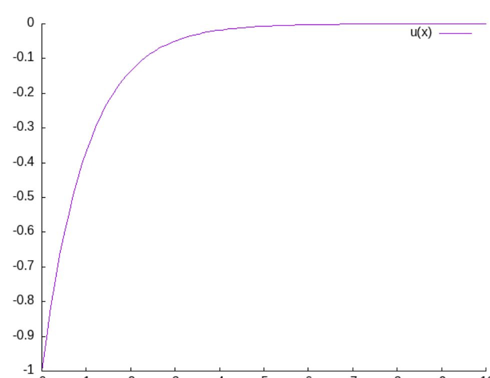


Figure 3: Exponential Utility Function $u(x) = a + be^{cx}$, $a = 0$, $b = -1$, $c = -1$

9.4 The common utility specifications.

Concave quadratic utility

$$u(x) = x - \frac{b}{2}x^2 \qquad b > 0$$

Negative exponential

$$u(x) = -e^{-bx} \qquad b > 0$$

Narrow power

$$u(x) = \frac{B}{B-1}x^{1-\frac{1}{B}} \qquad x > 0,\ B > 0$$

Extended power

$$u(x) = \frac{1}{B-1}(A + Bx)^{\left(1-\frac{1}{B}\right)} \qquad x > \max\left(-\frac{A}{B}, 0\right),\ B > 0\ A \neq 0$$

Sources and Further Reading Huang and Litzenberger (1988)

10 Stochastic Dominance

We now consider a way of choosing among assets by comparing their probability distributions.

Assumption 1 *Assets are defined along one dimension.*

The typical example of this monetary outcomes. Let X be the set of outcomes. For simplicity, X is assumed distributed according to the (continuous) distribution function $f()$. Let $F(y) = \int_{-\infty}^{y} f(x)dx$ be the *cumulative* probability distribution. An *asset* is fully described by its distribution function, either as f or F.

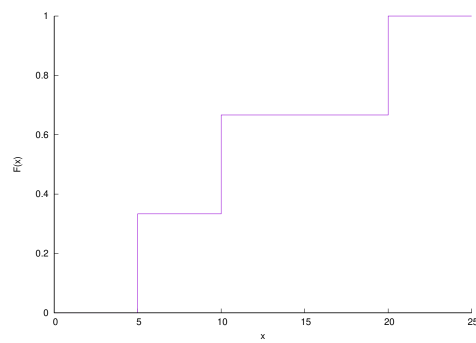
We assume the existence of an utility representation, summarized by a function $u(\cdot)$, such that the *expected utility*

$$E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

defines a ranking (Expected utility theorem).

Example

State	0	1	2
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Payoff	10	20	5



Question: By only making weak assumptions about the utility function, can we unambiguously say whether we prefer one asset to another?

Before looking at the stochastic case, consider the more basic notion of dominance.

Definition 9 (Dominance) *An asset A dominates an asset B if the payoff to asset A is always greater than or equal to the payoff to asset B.*

Exercise 5.

There are three possible states, 0, 1 and 2, and two assets A and B, with outcomes:

State	0	1	2
Asset A:	10	5	10
Asset B:	5	0	10

1. Can the two assets be ranked in some way?

Remark 1 *For dominance to apply, don't need agreement on probabilities, as long as all agree that all states have positive probabilities of occurring.*

Dominance is a very strong condition. In working markets one does not expect to observe dominated assets. Relaxing the notion of dominance leads to notions of *stochastic dominance*.

11 First order stochastic dominance

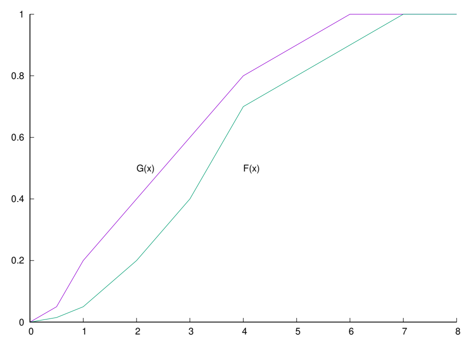
Definition 10 (First order stochastic dominance) *The distribution $F(\cdot)$ first order stochastically dominates the distribution $(G(\cdot)$ $(F \geq_{FSD} G))$ if for all $u : R \to R$, u bounded, nondecreasing:*

$$\int u(x)f(x)dx \geq \int u(x)g(x)dx$$

Proposition 5 *The distribution of monetary payoffs $F(\cdot)$ first order stochastically dominates the distribution $G(\cdot)$ iff $F(x) \leq G(x) \forall x$.*

Example

In the following figure, the distribution F dominates the distribution G by the criterion of first order stochastic dominance.



Remark 2 *Let x have the cdf F, and y have the cdf G. Suppose $F >_{FSD} G$. Then*

$$x = y + \alpha$$

where α is a positive random variable

Remark 3 *$A >_{FSD} B$ implies $E[A] > E[B]$, but $E[A] > E[B]$ does not imply $A >_{FSD} B$.*

Exercise 6.

Consider the following outcomes

	State	1	2	3
	probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Asset	A	10	5	20
	B	15	10	20
	C	10	20	5

1. Using the concept of dominance, can you rank some of these assets?
2. Using the concept of (first order) stochastic dominance, is it possible to choose one of these assets?

12 Second order stochastic dominance.

Definition 11 (Second order stochastic dominance) *For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, F second order stochastically dominates G if, for all bounded, nondecreasing, concave u, we have*

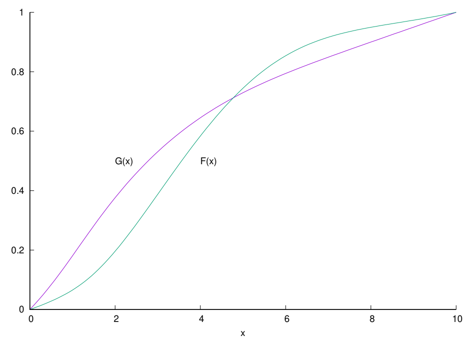
$$\int u(x)f(x)dx \geq \int u(x)g(x)dx$$

Notation: $F >_{SSD}$ G

Proposition 6 *$F >_{SSD} G$ if*

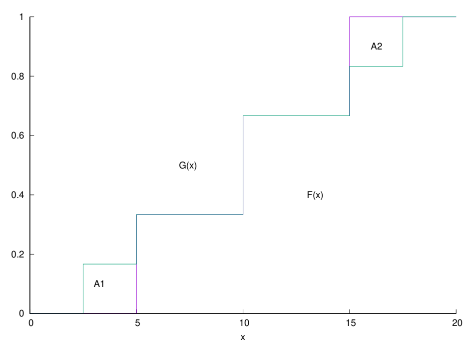
$$\int_{-\infty}^{x} G(t)dt \geq \int_{-\infty}^{x} F(t)dt \; \forall \; x$$

Example
In the following picture, the distribution F dominates the distribution G according to the criterion of second order stochastic dominance.



Remark 4 *Let $x \sim f$ and $y \sim g$. If $x >_{SSD} y$ then we can write $y = x + \epsilon$, where $E[\epsilon|x] = 0$. The random variable y is called a* mean preserving spread *of x.*

Example
As another illustration of second order stochastic dominance, consider the distrubution G, which has the dotted lines in the figure, and F with straight lines. $F >_{SSD} G$.



The area A_1 is equal to the area A_2.

Proposition 7 *If y is a mean preserving spread of x, then x is preferred to y by all concave utility functions $u(\cdot)$.*

Proposition 8 *If $x >_{SSD} y$, $var(x) \leq var(y)$*

Remark 5 *$var(x) < var(y)$ does not imply $x >_{SSD} y$.*

13 Further reading

Mas-Colell et al. (1995)

References

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