Estimation of The term structure of interest rates

Bernt Arne Ødegaard

18 April 2023

1 Intro

Term structure Two reasons for studying

- Term structure: pure "price of time" Economic interest, implications for theory.
- Modelling term structure: Pricing of derivative securities.

This lecture: How to estimate the term structure from market information.

2 Where do spot interest rates/discount factors come from?

How would you estimate the current interest rate?

Take an example: Treasury securities, for example those issued by the Norwegian government. Figure 1 shows the current bond prices (as of January 20, 2021) for the outstanding debt of the Norwegian state.

Figure 1 Norwegian treasury bond prices, January 20, 2021

EURONEXT Live markets — Navn, ISIN, ticker, indeks — Søk på nettstedet — NB

NAVN	UTSTEDER	MARKED	SLUTTDATO	KUPONGRENTE	SISTE	%	DATO/TID
NST474	NO0010572878	XOSL	2021-05-25	3.75%	% 101,731	-0,22%	30 Nov 2020
NST475	NO0010646813	XOSL	2023-05-24	2.0%	% 104,249	-0,04%	02 Dec 2020
NST476	NO0010705536	XOSL	2024-03-14	3.0%	% 108,76	0,03%	20 Nov 2020
NST477	NO0010732555	XOSL	2025-03-13	1.75%	% 105,37	-0,11%	04 Dec 2020
NST478	NO0010757925	XOSL	2026-02-19	1.5%	% 104,715	-0,41%	02 Dec 2020
NST479	NO0010786288	XOSL	2027-02-17	1.75%	% 106,665	0,07%	03 Dec 2020
NST480	NO0010821598	XOSL	2028-04-26	2.0%	% 109,285	-1,15%	30 Nov 2020
NST481	NO0010844079	XOSL	2029-09-06	1.75%	% 108,185	-0,04%	30 Nov 2020
NST482	NO0010875230	XOSL	2030-08-19	1.375%	% 105,025	0,25%	25 Nov 2020

The implied *interest rates* are found from the fact that the given *bond price* is the present value of the coupons and face value:

$$B_0 = \sum_{t=1}^{T} \frac{E[C_t]}{(1+r_t)^t} + \frac{E[F_T]}{(1+r_T)^T}$$

Alternatively, we can think in terms of *discount factors* d_t, where d_t is the current price of a promised payment of one dollar/NOK/... at a future date t

$$B_0 = \sum_{t=1}^{T} d_t E[C_t] + d_T E[F_T]$$

If we rewrite the expression involving spot rates above as

$$B_0 = \sum_{t=1}^{T} E[C_t] \left(\frac{1}{(1+r_t)^t} \right) + E[F_T] \left(\frac{1}{(1+r_T)^T} \right)$$

we see that the price of a zero coupon bond, the discount factor, is

$$d_t = \left(\frac{1}{(1+r_t)^t} \right)$$

Let us take a bit simpler example, showing how the estimation of the discount factors work.

Exercise 1.

A two-year Treasury bond with a face value of 1000 and an annual coupon payment of 8% sells for 982.50. A one-year T bill, with a face value of 100, and no coupons, sells for 90. Compounding is discrete, annual.

Given these market prices,

1. Find the implied one and two year interest rates.

3 Estimating the zero coupon term structure.

Problem: Do not observe zero coupon bonds for long maturities, only coupon bonds.

How to estimate the zero coupon term structure?

- Bootstrapping
- Nonlinear least squares estimation.

Exercise 2.

Consider the prices and terms of 6 bonds, where coupon is paid semiannually.

Principal	Time to maturity	Annual coupon	Bond price
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6
100	2.75	10	99.8

1. Based on these bond prices, estimate the term structure (with continous compounding) of interest rates by "bootstrapping."

4 Nonlinear least squares regression.

Another way to find the term structure. Assume the interest rate function $r(t, t+m)$ is a function of a few parameters. (These parameters will come from the choice of the functional form of the term structure.)

For any bond: If we know the interest rates, can find the price as

$$p(t,T) = \sum_i e^{-r(t,t_i)} C(t_i)$$

Where t_i is the date of coupon payment i.

If we have a large number of bond prices, could think of estimating the term structure by finding the functions $r()$ that solves the system of equations.

$$p_1(t,T) = \sum_i e^{-r(t,t_i)(t_i-t)} C_1(t_i)$$

$$p_2(t,T) = \sum_i e^{-r(t,t_i)(t_i-t)} C_2(t_i)$$

$\vdots$

$$p_n(t,T) = \sum_i e^{-r(t,t_i)(t_i-t} C_n(t_i)$$

If the term structure function $r(t,T)$ is identified by a few parameters, can not get a perfect fit.

Instead find the $r(t,T)$ that minimizes the error

$$\min_{r(\cdot)} \sum_{j=1}^{n} \left(p_j(t,T) - \sum_i e^{-r(t,t_i)(t_i-t)} C_j(t_i) \right)^2$$

Once we have decided on a model for the term structure, we can do this minimization.

Examples of term structure specifications:

Flat: $r(\cdot) = R$ One parameter, R, the interest rate.

Cubic spline: $e^{-r(t,T)(T-t)} = 1 + b\tau + c\tau^2 + d\tau^3 + \sum_k F_k(\tau - t_k) 1_{\{T<t_k\}}$, where $\tau = T - t$.

The functional form is not important for now, more interesting when actually using a term structure model like Ho-Lee, Cox-Ingersoll-Ross, Vasicek etc. We will get back to this when discussing these models.

See Green and Ødegaard (1997)

,

References

Zvi Bodie, Alex Kane, and Alan J Marcus. *Investments.* McGraw Hill/Irwin, 12 edition, 2021.

Richard C Green and Bernt Arne Ødegaard. Are there tax effects in the relative pricing of U.S. Government bonds? *Journal of Finance*, 52:609–633, June 1997.