Estimation of The term structure of interest rates

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18 April 2023

1 Intro

Term structure Two reasons for studying

- Term structure: pure "price of time" Economic interest, implications for theory.
- Modelling term structure: Pricing of derivative securities.

This lecture: How to estimate the term structure from market information.

2 Where do spot interest rates/discount factors come from?

How would you estimate the current interest rate?

Take an example: Treasury securities, for example those issued by the Norwegian government. Figure 1 shows the current bond prices (as of January 20, 2021) for the outstanding debt of the Norwegian state.

Figure 1 Norwegian treasury bond prices, January 20, 2021

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NAVN	UTSTEDER	MARKED	SLUTTDATO	KUPONGRENTE	SISTE	%	DATO/TID
NST474	NO0010572878	XOSL	2021-05-25	3.75%	% 101,731	-0,22%	30 Nov 2020
NST475	NO0010646813	XOSL	2023-05-24	2.0%	% 104,249	-0,04%	02 Dec 2020
NST476	NO0010705536	XOSL	2024-03-14	3.0%	% 108,76	0,03%	20 Nov 2020
NST477	NO0010732555	XOSL	2025-03-13	1.75%	% 105,37	-0,11%	04 Dec 2020
NST478	NO0010757925	XOSL	2026-02-19	1.5%	% 104,715	-0,41%	02 Dec 2020
NST479	NO0010786288	XOSL	2027-02-17	1.75%	% 106,665	0,07%	03 Dec 2020
NST480	NO0010821598	XOSL	2028-04-26	2.0%	% 109,285	-1,15%	30 Nov 2020
NST481	NO0010844079	XOSL	2029-09-06	1.75%	% 108,185	-0,04%	30 Nov 2020
NST482	NO0010875230	XOSL	2030-08-19	1.375%	% 105,025	0,25%	25 Nov 2020

The implied *interest rates* are found from the fact that the given *bond price* is the present value of the coupons and face value:

$$B_0 = \sum_{t=1}^{T} \frac{E[C_t]}{(1+r_t)^t} + \frac{E[F_T]}{(1+r_T)^T}$$

Alternatively, we can think in terms of *discount factors* d_t, where d_t is the current price of a promised payment of one dollar/NOK/... at a future date t

$$B_0 = \sum_{t=1}^{T} d_t E[C_t] + d_T E[F_T]$$

If we rewrite the expression involving spot rates above as

$$B_0 = \sum_{t=1}^{T} E[C_t] \left(\frac{1}{(1+r_t)^t} \right) + E[F_T] \left(\frac{1}{(1+r_T)^T} \right)$$

we see that the price of a zero coupon bond, the discount factor, is

$$d_t = \left(\frac{1}{(1+r_t)^t} \right)$$

Let us take a bit simpler example, showing how the estimation of the discount factors work.

Exercise 1.

A two-year Treasury bond with a face value of 1000 and an annual coupon payment of 8% sells for 982.50. A one-year T bill, with a face value of 100, and no coupons, sells for 90. Compounding is discrete, annual.

Given these market prices,

1. Find the implied one and two year interest rates.

Solution to Exercise 1.

To find the interest rates, first find the prices d_1 and d_2 of one dollar received respectively one and two years from now. These two discount factors will produce the current prices, and hence it satisfies the following set of equtions.

1. Discount factors (prices):

$$\begin{bmatrix} 982.50 = d_1 80 + d_2 1080 \\ 90 = d_1 100 \end{bmatrix}$$

Solving these equations we find prices d_1 and d_2

$$d_1 = \frac{90}{100} = 0.90$$

$$982.50 = 0.90 \times 80 + d_2 1080$$

$$d_2 = \frac{982.50 - 0.90 \times 80}{1080}$$

$$d_2 = 0.843$$

Summarizing

$$\begin{bmatrix} d_1 = 0.9 \\ d_2 = 0.84 \end{bmatrix}$$

Then, translate from discount factors to interest rates:

$$d_1 = \frac{1}{1+r_1}$$

$$r_1 = \frac{1}{d_1} - 1 = \frac{1}{0.9} - 1 = 0.11111 \approx 11\%$$

$$r_2 = \frac{1}{\sqrt{0.84}} - 1 = 0.09108945118 \approx 9\%$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \approx \begin{bmatrix} 11\% \\ 9\% \end{bmatrix}$$

For the technically interested (engineering students), these calculations is most compactly done in a matlab-like environment:

```
>> B=[982.50 90]
B =
   982.500    90.000
>> C=[80 1080;100 0]
C =
     80   1080
    100      0
>> d=inv(C)*B'
d =
   0.90000
   0.84306
>> r1=1/d(1)-1
r1 =  0.11111
>> r2=1/sqrt(d(2))-1
r2 =  0.089110
```

3 Estimating the zero coupon term structure.

Problem: Do not observe zero coupon bonds for long maturities, only coupon bonds.
How to estimate the zero coupon term structure?

- Bootstrapping
- Nonlinear least squares estimation.

Exercise 2.
Consider the prices and terms of 6 bonds, where coupon is paid semiannually.

Principal	Time to maturity	Annual coupon	Bond price
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6
100	2.75	10	99.8

1. Based on these bond prices, estimate the term structure (with continous compounding) of interest rates by "bootstrapping."

Solution to Exercise 2.
The first three do not pay any coupon, can find the spot rates immediately.

$$r(0.25) = \frac{-\ln(0.975)}{0.25} = 0.1013 = 10.13\%$$

$$r(0.5) = \frac{-\ln(0.949)}{0.5} = 0.1047 = 10.47\%$$

$$r(1.0) = \frac{-\ln(0.9)}{1} = 0.1054 = 10.54\%$$

Next, need to use a coupon bond to find $r(1.50)$.

Bond	time	0.5	1.0	1.5
	pays	4	4	4+100

Price = 96

$$96 = e^{-r(0.5)0.5}4 + e^{-r(1.0)}4 + e^{-r(1.5)1.5}104$$
$$96 = e^{-0.1013 \cdot 0.5}4 + e^{-0.1047}4 + e^{-r(1.5)1.5}104$$

Solve for $r(1.5)$:

$$96 = 3.8024 + 3.6024 + e^{-r(1.5)1.5}104$$
$$\frac{88.5952}{104} = e^{-r(1.5)1.5}$$
$$r(1.5) = \frac{-\ln(\frac{88.5952}{104})}{1.5} = 0.1069 = 10.69\%$$

Next, find the $t = 2$ - rate:

$$\begin{aligned} 101.6 &= e^{-r(0.5)0.5}6 + e^{-r(1.0)}6 + e^{-r(1.5)1.5}6 + e^{-r(2)2}106 \\ &= e^{-0.1047 \cdot 0.5}6 + e^{-0.1054}6 + e^{-0.1069 \cdot 1.5}6 + e^{-r(2)2}106 \end{aligned}$$

$$85.395 = e^{-r(2)2}106$$
$$\ln\left(\frac{85.395}{106}\right) = -r(2)2$$
$$-\frac{\ln\left(\frac{85.395}{106}\right)}{2} = r(2)$$
$$r(2) = 0.1081 = 10.81\%$$

After this, we have the following spot rates

t	$r(t)$
0.25	10.13
0.5	10.47
1.0	10.54
1.5	10.69
2.0	10.81

Now want to value a bond having payments

t	$r(t)$
0.25	5
0.75	5
1.25	5
1.75	5
2.25	5
2.75	5+100

How to get interest rates in between?

Answer: Linear interpolation

$$\begin{aligned} r(0.75) &= \frac{1}{2}r(0.5) + \frac{1}{2}r(1.0) = \frac{1}{2}0.1047 + \frac{1}{2}0.1054 = 0.10505 \\ r(1.25) &= \frac{1}{2}r(1.0) + \frac{1}{2}r(1.5) = \frac{1}{2}0.1054 + \frac{1}{2}0.1069 = 0.10615 \\ r(1.75) &= \frac{1}{2}r(1.5) + \frac{1}{2}r(2.0) = \frac{1}{2}0.1069 + \frac{1}{2}0.1081 = 0.1075 \end{aligned}$$

Problem: Do not have discount bond yields above 2, and need them for $t = 2.25$ and $t = 2.75$.

If $R = r(2.75)$ and $r(2) = 0.1081$, then would interpolate $r(2.25)$ as

$$r(2.25) = \frac{1}{3}r(2) + \frac{2}{3}R$$

Using this, can substitute for $r(2.25)$, and have only one equation in one unknown R, then solve for R.

$$\begin{aligned} & 99.88 \\ = \quad & e^{-r(0.25)0.25}5 + e^{-r(0.75)0.75}5 + e^{-r(1.25)1.25}5 + e^{-r(1.75)1.75}5 + e^{-r(2.25)2.25}5 + e^{-r(2.75)2.75}105 \\ = \quad & e^{-0.1013\cdot 0.25}5 + e^{-0.10505\cdot 0.75}5 + e^{-0.10615\cdot 1.25}5 + e^{-0.1075\cdot 1.75}5 + e^{-(0.1081\frac{1}{3}+R\frac{2}{3})2.25}5 + e^{-R\cdot 2.75}105 \\ = \quad & 4.8674 + 4.6212 + 4.3787 + 4.1426 + 4.616e^{-\frac{2}{3}R2.25}5 + e^{-R2.75}105 \end{aligned}$$

$$81.79 = 4.616e^{-\frac{2}{3}R2.25}5 + e^{-R2.75}105$$

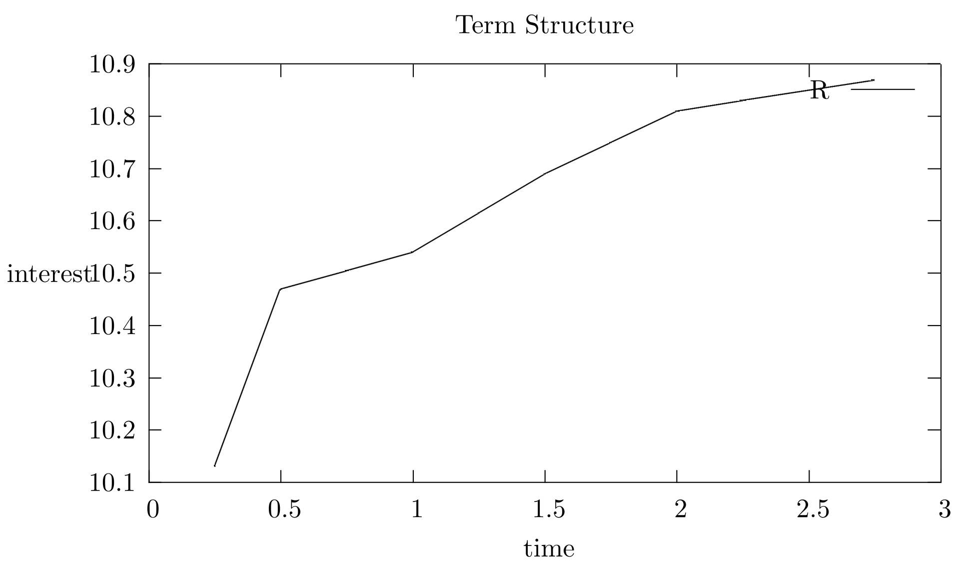
Solve for R by trial and error, get

$$R = 0.1087 = 10.87\%$$

End up with the following set of spot rates.

t	$r(t)$
0.25	10.13
0.5	10.47
0.75	10.505
1.0	10.54
1.25	10.615
1.5	10.69
1.75	10.75
2.0	10.81
2.25	10.83
2.75	10.87

Figure below shows the resulting term structure.



4 Nonlinear least squares regression.

Another way to find the term structure. Assume the interest rate function $r(t, t+m)$ is a function of a few parameters. (These parameters will come from the choice of the functional form of the term structure.)

For any bond: If we know the interest rates, can find the price as

$$p(t,T) = \sum_i e^{-r(t,t_i)} C(t_i)$$

Where t_i is the date of coupon payment i.

If we have a large number of bond prices, could think of estimating the term structure by finding the functions $r()$ that solves the system of equations.

$$p_1(t,T) = \sum_i e^{-r(t,t_i)(t_i-t)} C_1(t_i)$$

$$p_2(t,T) = \sum_i e^{-r(t,t_i)(t_i-t)} C_2(t_i)$$

$\vdots$

$$p_n(t,T) = \sum_i e^{-r(t,t_i)(t_i-t} C_n(t_i)$$

If the term structure function $r(t,T)$ is identified by a few parameters, can not get a perfect fit.

Instead find the $r(t,T)$ that minimizes the error

$$\min_{r(\cdot)} \sum_{j=1}^{n} \left(p_j(t,T) - \sum_i e^{-r(t,t_i)(t_i-t)} C_j(t_i) \right)^2$$

Once we have decided on a model for the term structure, we can do this minimization.

Examples of term structure specifications:

Flat: $r(\cdot) = R$ One parameter, R, the interest rate.

Cubic spline: $e^{-r(t,T)(T-t)} = 1 + b\tau + c\tau^2 + d\tau^3 + \sum_k F_k(\tau - t_k) 1_{\{T<t_k\}}$, where $\tau = T - t$.

The functional form is not important for now, more interesting when actually using a term structure model like Ho-Lee, Cox-Ingersoll-Ross, Vasicek etc. We will get back to this when discussing these models.

See Green and Ødegaard (1997)

5 Summary – term structure estimation

Term structure *implied* in prices of fixed income structure

Estimation: What is the *term structure curve* that best explains the data?

- Bootstrapping – from the shortest duration securities, work your way up
- Nonlinear least squares estimation

References

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