PROBLEM SET: Mean Variance

Exercise 1.

Arnold [2]

Arnold is tired of the commando business. He wants to settle down in some small cabin and live off his savings. He has evenly divided his money in two stocks, General Mills and Bristol Myers. On past evidence both stocks have a standard deviation of 20%.

1. Suppose the correlation between the two stocks is 1. What is the standard deviation of Arnold's portfolio?
2. Suppose now the correlation is zero. What is now the standard deviation of his portfolio?

Exercise 2.

Arnold the Barbarian [5]

Arnold the Barbarian can invest in two assets, a sword-making factory or an ale-brewery. He thinks the probability distributions for returns of the two assets 1 and 2 are:

r_1	Probability (r_1)	r_2	Probability (r_2)
10%	0.1	22%	0.2
25	0.4	23	0.5
30	0.3	24	0.2
32	0.2	50	0.1

1. Suppose Arnold can invest in either 1 or 2. Suppose also that Arnold chooses on the basis of mean-variance efficiency. Which asset should Arnold choose?
2. Assume the sword-making business is independent of the ale-brewing business. In a graph plot the possible combinations of expected return and standard deviation Arnold can achieve if he can invest in any portfolio of the two assets.
3. Suppose now that sword-making is perfectly negatively correlated with ale-brewing. Now plot all possible combinations of expected return and standard deviation Arnold can achieve if he can invest in any portfolio of the two assets.

Exercise 3.

portfolios (BM 7.14) [3]

Here are some historical data on the risk characteristics of MCI and Polaroid:

	MCI	Polaroid
$\beta(beta)$	1.24	0.99
Yearly st. dev. of return	34	28

The standard deviation of the return on the market was 20 percent.

1. Assume the correlation coefficient of MCI's return versus Polaroid's is 0.20. What is the standard deviation of a portfolio half invested in Polaroid and half of MCI?
2. What is the standard deviation of a portfolio one-third invested in MCI, one-third in Polaroid, and one-third in Treasury bills?
3. What is the standard deviation if the portfolio is split equally between MCI and Polaroid and financed at 50 percent margin - that is, the investor puts up only 50 percent of the total amount and borrows the balance from the broker?

4. What is the *approximate* standard deviation of a portfolio composed of 100 stocks with β's of 1.24, like MCI? How about 100 stocks like Polaroid? (*Hint*: The last part should not require anything but the simplest arithmetic to answer.)

Exercise 4.

Return (RWJ 9.8) [2]

The probability that the economy will experience moderate growth next year is 0.4 The probability of a recession is 0.3, and the probability of a rapid expansion is also 0.3 If the economy falls into a recession, you can expect to receive a return on your portfolio of 2%. With moderate growth your return will be 5%. If there is a rapid expansion, your portfolio will return 15%.

1. What is your expected return?
2. What is the standard deviation of that return?

Exercise 5.

Suppose you hold a portfolio of two stocks. The relevant information on these two stocks is given below.

Stock	Weight	Variance	Expected return
1	0.6	0.04	0.12
2	0.4	0.09	0.20

1. Compute the expected return and variance of your portfolio assuming ρ_{12} is 0, -1 and 1.
2. Show that the portfolio weights on stock 1 that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_2^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

   What are the minimal variances for teh three possible values of correlation

3. In a diagram illustrate the possible combinations of means and returns for the three possible values of the correlation.
4. Suppose that stocks 1 and 2 are the only securities in the market. What combinations of the two stocks would investors be willing to hold in their portfolios?

Exercise 6.

MSR & HFF (BM 7.15) [3]

You are analysing the stocks of the two companies Macbeth Spot Removers (Purveyors of cleaning services to the Scottish Nobility) (stock A) and Hamlet Fine Foods (Serving the Danish court) (stock B). You believe there is a 40% chance that stock A will decline by 10% and a 60% chance that it will increase by 20%. Correspondingly, there is a 30% chance that stock B will decline by 10% and a 70% chance it will rise by 20%. The correlation coefficient between the two stocks is 0.7.

- Calculate the expected return, the variance, and the standard deviation for each stock.
- Then calculate the covariance between their return.

Exercise 7.

[3]

You are considering investing your wealth in two stocks, A and B. Stock A has a current price of kr 100, an expected return of 10% and a standard deviation of 10%. Stock B has a current price of kr 50, an expected return of 12% and a standard deviation of 10%. You don't know the covariance between the two shares. To avoid excessive costs you only buy round lots of 100 shares. You are therefore limited to the following portfolios: Portfolio I: 200 A shares, 0 B shares. Portfolio II: 100 A shares, 200 B shares. Portfolio III: 0 A shares, 400 B shares. You are making decisions based on mean-variance optimization. Which of the following statements is correct?

(a) Only portfolio I is a possibly optimal portfolio.

(b) Only portfolio III is a possibly optimal portfolio.

(c) Both portfolios I and II are possibly optimal portfolios.

(d) Both portfolios II and III are possibly optimal portfolios.

(e) I choose not to answer.

Exercise 8.

[3]
Consider a portfolio that is made up of the risk-free asset and a risky asset. The return for the risk free asset is 13%. The expected return for the risky asset is 18%. The standard deviation of the risky asset is 23%. You invest 33% of your wealth in the risk-free asset. What is the expected return for your portfolio?

(a) 13.00

(b) 14.67

(c) 16.33

(d) 19.67

(e) I choose not to answer.

Exercise 9.

[3]
Consider a portfolio that is made up of the riskfree asset and a risky asset. The return on the risk-free asset is 13%. The expected return for the risky asset 18%. The standard deviation of the risky asset is 23%. You invest one third of your portfolio in the risky asset. What is the standard deviation of the riskfree asset?

(a) 0.00

(b) 0.18

(c) 0.23

(d) There is not enough information to answer the question.

(e) I choose not to answer.

Exercise 10.

[2]
You are trading in a market that has only two securities available. Security A has an expected return of 10% and a standard deviation of 30%. Security B has an expected return of 15% and a standard deviation of 110%. You place 30% of your money in A and 70% in B. What is the expected return on your portfolio?

1. 10.0%

2. 12.5%
3. 13.5%
4. 15.0%
5. I choose not to answer.

Empirical

Solutions

PROBLEM SET: Mean Variance

Solution to Exercise 1.

Arnold [2]

1. Standard deviation when $\rho_{12} = 1$: In this case the SD is a linear combination of the two, so it is 20%. But let us calculate the SD just to make sure

$$
\begin{aligned}
\sigma_p^2 &= \left(\tfrac{1}{2}\right)^2 \sigma_1^2 + \left(\tfrac{1}{2}\right)^2 \sigma_2^2 \\
&\quad +2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sigma_{12} \\
&= \left(\tfrac{1}{2}\right)^2 \sigma_1^2 + \left(\tfrac{1}{2}\right)^2 \sigma_2^2 \\
&\quad +2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sigma_1\sigma_2\rho_{12} \\
&= \left(\tfrac{1}{2}\right)^2 (0.2)^2 + \left(\tfrac{1}{2}\right)^2 (0.2)^2 \\
&\quad +2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) 0.2 \cdot 0.2 \\
&= 4\left(\tfrac{1}{2}\right)^2 (0.2)^2 \\
&= 4\tfrac{1}{4}(0.2)^2 \\
&= (0.2)^2 \\
\sigma_p &= \sqrt{(0.2)^2} \\
&= 0.2
\end{aligned}
$$

2. Standard deviation when $\rho_{12} = 0$:

$$
\begin{aligned}
\sigma_p^2 &= \left(\tfrac{1}{2}\right)^2 \sigma_1^2 + \left(\tfrac{1}{2}\right)^2 \sigma_2^2 \\
&\quad +2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sigma_1\sigma_2\rho_{12} \\
&= \left(\tfrac{1}{2}\right)^2 (0.2)^2 + \left(\tfrac{1}{2}\right)^2 (0.2)^2 \\
&= 2\left(\tfrac{1}{2}\right)^2 (0.2)^2 \\
&= 2\tfrac{1}{4}(0.2)^2 \\
&= \tfrac{1}{2}(0.2)^2 \\
\sigma_p &= \sqrt{\tfrac{1}{2}(0.2)^2} \\
&= \sqrt{\tfrac{1}{2}}0.2 \\
&= 0.141
\end{aligned}
$$

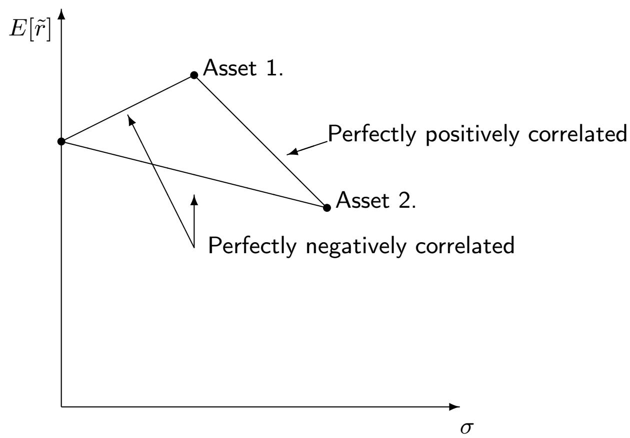
Solution to Exercise 2.

Arnold the Barbarian [5]

r_1	Probability (r_1)	r_2	Probability (r_2)
10%	0.1	22%	0.2
25	0.4	23	0.5
30	0.3	24	0.2
32	0.2	50	0.1

$$
\begin{aligned}
E[\tilde{r}_1] &= 26.4\% \\
\sigma(\tilde{r}_1) &= 6.15\% \\
E[\tilde{r}_2] &= 25.7\% \\
\sigma(\tilde{r}_2) &= 8.12\%
\end{aligned}
$$

Since Asset 2 has lower expected return *and* higher variance than asset 1, Arnold should choose asset 1.



Solution to Exercise 3.

portfolios (BM 7.14) [3]

1. Variance $=\frac{1}{2}^2\text{var}(r_{MCI}) + 2(\frac{1}{2})(\frac{1}{2})\text{cov}(r_{MCI}, r_{Pol}) + \frac{1}{2}^2\text{var}(r_{Pol})$ Standard deviation = Square root of variance
2. Variance $=\frac{1}{3}^2\text{var}(r_{MCI}) + 2(\frac{1}{3})(\frac{1}{3})\text{cov}(r_{MCI}, r_{Pol}) + \frac{1}{3}^2\text{var}(r_{Pol})$
3. Variance$=(-\frac{1}{2}) \cdot 0 + \frac{3}{4}^2\text{var}(r_{MCI}) + 2(\frac{3}{4})(\frac{3}{4})\text{cov}(r_{MCI}, r_{Pol}) + \frac{3}{4}^2\text{var}(r_{Pol})$
4.

Solution to Exercise 4.

Return (RWJ 9.8) [2]

1. Expected Return.
$$E[r] = 0.3 \cdot 2\% + 0.4 \cdot 5\% + 0.3 \cdot 15\% = 7.1\%$$
2. Variance.
$$\begin{aligned}
\sigma^2 &= E\left[(r - E[r])^2\right] \\
&= 0.3 \cdot (0.02 - 0.071)^2 \\
&\quad +0.4 \cdot (0.05 - 0.071)^2 \\
&\quad +0.3 \cdot (0.15 - 0.071)^2 \\
&= 0.3 \cdot 0.002601 \\
&\quad +0.4 \cdot 0.000441 \\
&\quad +0.3 \cdot 0.006241 \\
&= 0.002829 \\
\sigma &= \sqrt{0.002829} = 0.053 = 5.3\%
\end{aligned}$$

Solution to Exercise 5.

1. The expected return on the portfolio is
$$E[\tilde{r}_p] = 0.6 \cdot 0.12 + 0.4 \cdot 0.20 = 0.152$$

The variance of your portfolio will depend on the correlation coefficient ρ_{12}. Suppose $\rho_{12} = 0$

$$\begin{aligned} \sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \\ &= 0.6^2 \cdot 0.4 + 0.4^2 \cdot 0.09 = 0.0288 \end{aligned}$$

$$\sigma_p = \sqrt{0.0288} = 0.1697 = 16.97\%$$

Note that σ_p^2 is smaller than both σ_1^2 and σ_2^2. Why? Suppose $\rho_{12} = 1$

$$\begin{aligned} \sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2 \cdot \omega_1\omega_2\sigma_1\sigma_2 \\ &= (\omega_1\sigma_1 + \omega_2\sigma_2)^2 \\ &= (0.6 \cdot 0.2 + 0.4 \cdot 0.3)^2 \\ &= 0.0576 \end{aligned}$$

$$\sigma_p = \sqrt{0.0576} = 0.24 = 24\%$$

Note that when $\rho_{12} = 1$, the standard deviation of the portfolio is equal to the weighted average of the standard deviations for each of the stocks in the portfolio.

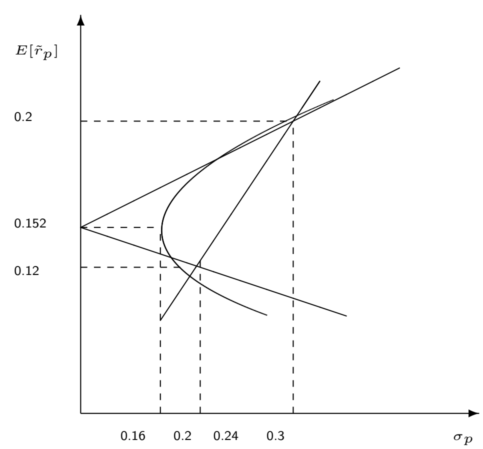
$$\sigma_p = \omega_1\sigma_1 + \omega_2\sigma_2$$

when $\rho_{12} = 1$. Suppose $\rho_{12} = -1$

$$\begin{aligned} \sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 - 2 \cdot \omega_1\omega_2\sigma_1\sigma_2 \\ &= (\omega_1\sigma_1 - \omega_2\sigma_2)^2 \\ &= (0.6 \cdot 0.2 + 0.4 \cdot 0.3)^2 = 0 \end{aligned}$$

$$\sigma_p = \sqrt{0} = 0\%$$

2. The diagram below shows the locus of portfolios you could create from stocks 1 and 2 for three different values of ρ_{12}.



3. Note that when $\rho_{12} = -1$, the standard deviation of the portfolio is equal to zero if the portfolio weight on the first stock is chosen as

$$\omega_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

Since $\omega_1 + \omega_2 = 1$, the portfolio weight on the second stock is

$$\omega_2 = 1 - \omega_1 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

Note that the portfolio weights on stock 1 that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_2^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

4.

Correlation ρ_{12}	ω_1^*	$E[\tilde{r}_p]$	σ_p
-1	0.60	15.2%	0
0	0.69	14.5%	16.64 %
1	3.00	-4.0%	0

Solution to Exercise 6.

MSR & HFF (BM 7.15) [3]

$$\begin{array}{ccc} A & p & \tilde{r}_A \\ & 0.4 & -10\% \\ & 0.6 & 20\% \\ B & p & \tilde{r}_B \\ & 0.3 & -10\% \\ & 0.7 & 20\% \\ & \rho_{AB} = 0.7 & \end{array}$$

$$\begin{aligned} E[\tilde{r}_A] &= 0.4(-0.10) + 0.6(0.20) \\ &= 0.08 \\ &= 8\%. \\ \sigma_A^2 &= E[(r_A - E[r_A])^2] \\ &= 0.4(-0.10 - 0.08)^2 \\ &\quad +0.6(0.20 - 0.08)^2 \\ &= 0.0216 \\ \sigma_A &= \sqrt{\sigma_A^2} \\ &= 14.7\% \\ E[\tilde{r}_B] &= 0.3(-0.10) + 0.7 \cdot 0.2 \\ &= 11\% \\ \sigma_B^2 &= 0.0189 \\ \sigma_B &= 13.7\% \\ \text{cov}(\tilde{r}_A, \tilde{r}_B) &= \rho_{AB} \cdot \sigma_A \sigma_B \\ &= 0.7 \cdot 0.147 \cdot 0.137 \\ &= 0.0141 \end{aligned}$$

Solution to Exercise 7.

[3]

Stock A is dominated by stock B, since it has a higher return and the same standard deviation. Hence portfolio I can not be optimal, ruling out (a) and (c). Since we don't know the covariance we can not rule out that the portfolio combining A and B having lower variance than stock B only, hence both II and III are potentially optimal. (d) is correct.

Solution to Exercise 8.

[3]

$$\frac{1}{3}r_f + \frac{2}{3}\tilde{r} = \frac{1}{3}0.13 + \frac{2}{3}0.23 = 16.33\%$$

(c) is correct

Solution to Exercise 9.

[3]

0.00 (a) is correct

Solution to Exercise 10.

[2]

$$0.3 \times 0.1 + 0.7 \times 0.15 = 13.5$$

(c) is correct.