Mean Variance

Lecture overview:

- Why only care about means and variances?
- The mechanics of mean-variance calculations
- Implications for feasible portfolios
  - - minimum variance set
  - - efficient set
- Equilibrium consequence: CAPM
- Using the CAPM

Portfolio opportunities

Working towards a way of quantifying risk, the first step to price it.
Will do so in the context of a *portfolio* problem.
Personal portfolio selection
Tradeoff expected return and risk

The starting point for formulating this decision problem is to simplify it into a tradeoff between

- Increase in value of a portfolio
    - Measured by expected return
- The variability of return
    - Measured by standard deviation of returns

Boils down to assuming investors have preferences defined over means ($E[r]$) and variances $\sigma^2(r)$.

$$U(p) = U(E[r_p], \sigma^2(r_p))$$

Want more returns

$$\frac{\ell U}{\ell E[r]} > 0$$

Dislike risk

$$\frac{\ell U}{\ell \sigma^2(r)} < 0$$

follows straightforward from calculation of means and variances of portfolios.
We therefore start there, with that calculation.

Basic tools: Expectation, Variance, Covariance.

Reminder of statistical results

Expectation

Given a set of possible states
the expected value is the sum of probabilities times outcomes.
In the case where we have only two possible outcomes, $X = X_1$ or $X = X_2$
define the expectation as

$$E[X] = P(X_1) \cdot X_1 + P(X_2) \cdot X_2$$

where

- $P(X_1)$ is the probability that outcome 1 will happen, and
- $P(X_2)$ is the probability that outcome 2 will happen.

Exercise

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the expected value next period for each of the three investment opportunities

Exercise solution

Expected values

$$
\begin{aligned}
E[\text{Bond}] &= \tfrac{1}{2}100 + \tfrac{1}{2}100 \\
&= 100 \\
E[\text{Stock 1}] &= \tfrac{1}{2}120 + \tfrac{1}{2}80 \\
&= 100 \\
E[\text{Stock 2}] &= \tfrac{1}{2}40 + \tfrac{1}{2}160 \\
&= 100
\end{aligned}
$$

Variance

The variance of a random variable X is defined as

$$\mathrm{var}(X) = \sigma^2(X) = E\left[(X - E[X])^2\right]$$

Exercise

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the variance of the expected value for each of the three investment opportunities
2. Calculate the standard deviation of this value for each of the three assets.

Exercise

Variances

$$\begin{aligned}
\text{var(Bond)} &= \tfrac{1}{2}(100-100)^2 + \tfrac{1}{2}(100-100)^2 \\
&= 0 \\
\text{var(Stock 1)} &= \tfrac{1}{2}(120-100)^2 + \tfrac{1}{2}(80-100)^2 \\
&= 400 \\
\text{var(Stock 2)} &= \tfrac{1}{2}(40-100)^2 + \tfrac{1}{2}(160-100)^2 \\
&= 3600
\end{aligned}$$

Exercise

Standard deviation

$$
\begin{aligned}
SD(\text{Bond}) &= \sqrt{0} &= 0 \\
SD(\text{Stock 1}) &= \sqrt{400} &= 20 \\
SD(\text{Stock 2}) &= \sqrt{3600} &= 60
\end{aligned}
$$

some common rules

- Variances are not, in general, additive:

$$\begin{aligned} &\text{var}(\tilde{X} + \tilde{Y}) \\ &\quad = \text{var}(\tilde{X}) + 2\text{cov}(\tilde{X}, \tilde{Y}) + \text{var}(Y) \end{aligned}$$

- The variance of a constant *a* times a random variable X:

$$\text{var}(a\tilde{X}) = a^2\text{var}(\tilde{X})$$

Standard deviation: measure of dispersion that is easier to interpret defined as the square root of the variance.

$$\sigma(X) = \sqrt{\mathsf{var}(X)}$$

Covariance.

$$\begin{aligned} \text{cov}(X, Y) &= \sigma_{X,Y} \\ &= E\left[(X - E[X])(Y - E[Y])\right] \end{aligned}$$

The *covariance* $\sigma_{X,Y}$ is a measure of the degree to which the two variables move together.

the covariance of a random variable with itself is the variance:

$$\begin{aligned} \mathrm{cov}(X, X) &= E\left[(X - E[X])(X - E[X])\right] \\ &= E\left[(X - E[X])^2\right] \\ &= \mathrm{var}(X) \end{aligned}$$

Some important properties of covariances.

- Covariances are additive.

$$\text{cov}(\tilde{X} + \tilde{Y}, \tilde{Z}) = \text{cov}(\tilde{X}, \tilde{Z}) + \text{cov}(\tilde{Y}, \tilde{Z})$$

- Covariance of a constant with a random variable is zero.

$$\text{cov}(c, \tilde{X}) = 0$$

- Constants multiplying random variables can be factored outside the covariance.

$$\text{cov}(c \cdot \tilde{X}, b \cdot \tilde{Y}) = c \cdot b \cdot \text{cov}(\tilde{X}, \tilde{Y})$$

- Putting it all together.

$$\begin{aligned} &\text{cov}(c\tilde{X} + b\tilde{Y} - d, \tilde{Z}) \\ &\quad = \quad c \cdot \text{cov}(\tilde{X}, \tilde{Z}) + b \cdot \text{cov}(\tilde{Y}, \tilde{Z}) \end{aligned}$$

What is $\text{cov}(\tilde{X} + \tilde{Y}, \tilde{W} + \tilde{Z})$?

Relative measure of how much two variables covary.

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sigma(X)\sigma(Y)}$$

$$-1 \leq \rho \leq 1$$

$\rho = 1$	Perfect positive correlation
$1 > \rho > 0$	Positive correlation
$\rho = 0$	Uncorrelated
$0 > \rho > -1$	Negative correlation
$\rho = -1$	Perfect negative correlation

The covariance between two variables X and Y in terms of the correlation coefficient:

$$\text{cov}(\tilde{X}, \tilde{Y}) = \sigma(\tilde{X})\sigma(\tilde{Y})\rho_{X,Y}$$

Exercise

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the covariance of the values next period for the two stocks
2. Calculate the correlation between the two stocks.

Exercise solution

Covariance

$$\begin{aligned}
&\text{cov}(S_1, S_2) \\
&\quad = \tfrac{1}{2}(40 - 100)(120 - 100) \\
&\qquad + \tfrac{1}{2}(160 - 100)(80 - 100) \\
&\quad = \tfrac{1}{2}(-60)(20) + \tfrac{1}{2}(60)(-20) \\
&\quad = -1200
\end{aligned}$$

Exercise solution

Correlation

$$\rho_{1,2} = \frac{-1200}{60 \cdot 20} = -1$$

Which is perfect negative correlation.

The mean variance paradigm for quantifying risk

The mechanics of creating mean variance optimal portfolios
Basic idea: Construct portfolios of securities that offer the highest *expected return* for a given level of *risk*, where risk is measured by the variance/standard deviation of portfolio returns.

Measuring Portfolio returns

The return on a portfolio of securities, $\tilde{r}_p$, is a weighted average of the returns on the individual securities making up the portfolio.

$$\tilde{r}_p = \sum_{j=1}^{N} \omega_j \tilde{r}_j$$

ω_j: proportion of the portfolio invested in security j

The *expected* return on the portfolio is

$$\begin{aligned} E[\tilde{r}_j] &= E\left[\sum_{j=1}^{N} \omega_j \tilde{r}_j\right] \\ &= \sum_{j=1}^{N} \omega_j E[\tilde{r}_j] \end{aligned}$$

Measuring portfolio risk

Risk is a difficult concept.
We *operationalize* it by the *dispersion* of possible outcomes.
The tighter the probability distribution of outcomes, the smaller is the risk of the investment.
The standard statistical measures of dispersion are *variance* and *standard deviation*.
The variance of the rate of return on a portfolio, σ_p^2

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

The simplest case, choose between two assets.

Variance of two-asset portfolio

$$
\begin{aligned}
&= \sum_{j=1}^{2}\sum_{j=1}^{2}\omega_j\omega_i\sigma_{ij} \\
&= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12}
\end{aligned}
$$

Exercise

Suppose you hold a portfolio of two stocks.

Stock	Weight	Variance	Expected return
1	0.6	0.04	0.12
2	0.4	0.09	0.20

1. Compute the expected return and variance of your portfolio assuming ρ_{12} is 0, -1 and 1.
2. Sketch how the portfolio variance and expectation would vary for these three cases.
3. Find the set of portfolio weights that minimizes the portfolio variance

Exercise solution

The expected return on the portfolio is

$$E[\tilde{r}_p] = 0.6 \cdot 0.12 + 0.4 \cdot 0.20 = 0.152$$

The variance of your portfolio will depend on the correlation coefficient ρ_{12}.
$\rho_{12} = 0$

$$\begin{aligned} \sigma_p^2 &= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 \\ &= 0.6^2 \cdot 0.4 + 0.4^2 \cdot 0.09 = 0.0288 \end{aligned}$$

$$\sigma_p = \sqrt{0.0288} = 0.1697 = 16.97\%$$

Note that σ_p^2 is smaller than both σ_1^2 and σ_2^2. Why?

Exercise solution

m $\rho_{12} = 1$

$$\begin{aligned}\sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\cdot\omega_1\omega_2\sigma_1\sigma_2 \\ &= (\omega_1\sigma_1 + \omega_2\sigma_2)^2 \\ &= (0.6\cdot 0.2 + 0.4\cdot 0.3)^2 \\ &= 0.0576\end{aligned}$$

$$\sigma_p = \sqrt{0.0576} = 0.24 = 24\%$$

Exercise solution

Note that when $\rho_{12} = 1$, the standard deviation of the portfolio is equal to the weighted average of the standard deviations for each of the stocks in the portfolio.

$$\sigma_p = \omega_1 \sigma_1 + \omega_2 \sigma_2$$

when $\rho_{12} = 1$.

Exercise solution

$\rho_{12} = -1$

$$\begin{aligned}\sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 - 2\cdot\omega_1\omega_2\sigma_1\sigma_2 \\ &= (\omega_1\sigma_1 - \omega_2\sigma_2)^2 \\ &= (0.6\cdot 0.2 + 0.4\cdot 0.3)^2 = 0\end{aligned}$$

$$\sigma_p = \sqrt{0} = 0\%$$

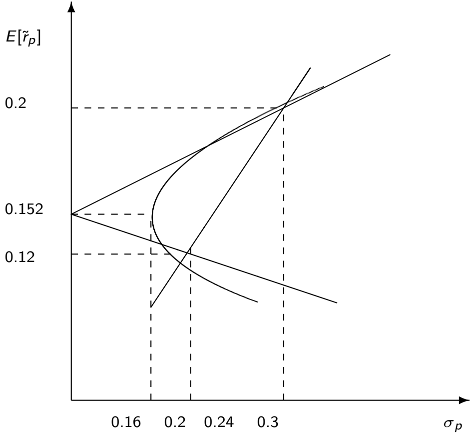
Note that when $\rho_{12} = -1$, the standard deviation of the portfolio is equal to zero if the portfolio weight on the first stock is chosen as

$$\omega_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

Exercise solution

Since $\omega_1 + \omega_2 = 1$, the portfolio weight on the second stock is

$$\omega_2 = 1 - \omega_1 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

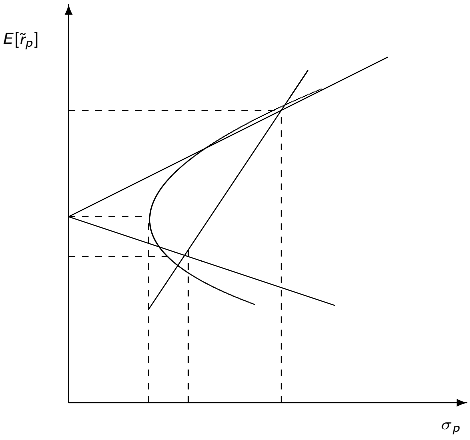


Exercise solution

Note that the portfolio weights on stock 1 that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

Correlation ρ_{12}	ω_1^*	$E[\tilde{r}_p]$	σ_p
-1	0.60	15.2%	0
0	0.69	14.5%	16.64 %
1	3.00	-4.0%	0


$E[\tilde{r}_p]$
σ_p

The portfolio weights on stock 1 in a two-stock problem that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

General Case

Go back to the general case

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

Substituting for $\tilde{r}_p$ and $E[\tilde{r}_p]$:

$$\begin{aligned}
\sigma_p^2 &= E\left[\left(\sum_{j=1}^{N}\omega_j\tilde{r}_j - \sum_{j=1}^{N}\omega_j E[\tilde{r}_j]\right)^2\right] \\
&= E\left[\left(\sum_{j=1}^{N}\omega_j(\tilde{r}_j - E[\tilde{r}_j])\right)^2\right] \\
&= E\left[\sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])\right] \\
&= \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])]
\end{aligned}$$

But notice that these expectations are

$$
\begin{aligned}
& E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])] \\
& \quad = \quad \text{cov}(\tilde{r}_j, \tilde{r}_i) \\
& \quad = \quad \sigma_{ji}.
\end{aligned}
$$

Therefore, we can write the variance of a portfolio as

$$\sigma_p^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} \omega_j \omega_i \sigma_{ij}$$

The *covariance* σ_{ij} is a measure of the degree to which the returns on securities i and j move together. Intuitively, if the returns on securities i and j tend to move in the same direction, the covariance will be positive. If the returns move in opposite directions, the covariance will be negative.

Correlation

Another measure of the degree to which the returns on securities i and j move together is the *correlation coefficient* ρ_{ij}. The correlation coefficient is computed as

$$\begin{aligned}\rho_{ij} &= \frac{\sigma_{ij}}{\sigma_i \sigma_j} \\ &= \frac{\text{cov}(\tilde{r}_i, \tilde{r}_j)}{\sqrt{\text{var}(\tilde{r}_i)}\sqrt{\text{var}(\tilde{r}_j)}} \\ &= \frac{\text{cov}(\tilde{r}_i, \tilde{r}_j)}{SD(\tilde{r}_i)SD(\tilde{r}_j)}\end{aligned}$$

The correlation coefficient, unlike the covariance, will always take on values between -1 and 1. That is $-1 \leq \rho_{ij} \leq 1$.

Correlation

The returns on securities i and j are said to be *positively* correlated if $\rho_{ij} > 0$, *negatively* correlated if $\rho_{ij} < 0$ and *uncorrelated* if $\rho_{ij} = 0$. A special case of positive correlation is when $\rho_{ij} = 1$. Then we say $\tilde{r}_i$ and $\tilde{r}_j$ are perfectly positively correlated. On the other hand if $\rho_{ij} = -1$ we say $\tilde{r}_i$ and $\tilde{r}_j$ are perfectly negatively correlated. The correlation coefficient has another interesting interpretation. The square of the correlation coefficient is the R^2 of the regression of the returns for security i on the returns for security j. In other words, the correlation coefficient tells us something about the extent to which the returns on one security can *explain* the returns on another security.

Portfolio

three alternative ways to write the variance of a portfolio.

$$\begin{aligned}\sigma_p^2 &= \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i\sigma_{ij} \\ &= \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i\rho_{ij}\sigma_j\sigma_i \\ &= \sum_{j=1}^{N}\omega_j\sigma_{jp}\end{aligned}$$

The last line of the above equation comes from the fact that

$$\sum_{i=1}^{N}\omega_i\sigma_{ji} = \text{cov}\left(\tilde{r}_j, \sum_{i=1}^{N}\omega_i\tilde{r}_i\right) = \sigma_{jp}$$

Thus, the covariance of a portfolio is equal to the weighted average of the covariances between the return on the portfolio and the returns on the individual securities in the portfolio.
The contribution of security j to the variance of a portfolio is $\omega_j\sigma_{jp}$.

Decomposing portfolio risk

To better understand the role of diversification and the limits to which diversification can reduce risk, let's recall the basic formula for a portfolio consisting of N securities.

$$
\begin{aligned}
\sigma_p^2 &= \sum_{j=1}^{N} \sum_{i=1}^{N} \omega_i \omega_j \sigma_{ji} \\
&= \sum_{j=1}^{N} \omega_j^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{i \neq j}^{N} \omega_j \omega_i \sigma_{ji}
\end{aligned}
$$

The first term represent the contribution of the securities *own variance* to the variance of the portfolio.

The second term represents the contribution of the *covariances* between securities to the variance of the portfolio.

Note that there are N terms in the first summation (involving variances) and $N \cdot (N - 1)$ terms in the second summation (involving covariances).

Decomposing portfolio risk

A *well-diversified* portfolio is one where the contribution of the own variance to the variance of the portfolio is negligible, or close to zero.
To illustrate:
consider a portfolio that places a constant weight $\omega_j = 1/N$ for all j, on each asset. We call this kind of portfolio an *equally weighted portfolio*. The variance of an equally-weighted portfolio is

$$\sigma_p^2 - \sum_{j=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{i \neq j}^{N} \left(\frac{1}{N}\right)^2 \sigma_{ji}$$

Decomposing portfolio risk

The first term can be rewritten as

$$\begin{aligned} \sum_{j=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_j^2 &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{N}\sigma_j^2 \\ &= \frac{1}{N}\overline{\sigma}^2 \end{aligned}$$

where $\overline{\sigma}^2$ is the *average variance*.

Decomposing portfolio risk

The second term can be written as

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{i\neq j}^{N}\left(\frac{1}{N}\right)^2\sigma_{ij} \\
&\quad= \frac{N-1}{N^2}\sum_{j=1}^{N}\sum_{i\neq j}\frac{\sigma_{ji}}{N-1} \\
&\quad= \frac{N-1}{N}\sum_{j=1}^{N}\frac{1}{N}\begin{pmatrix}\text{Avg. covar.}\\ \text{with asset } j\end{pmatrix} \\
&\quad= \frac{N-1}{N}\begin{pmatrix}\text{Avg. covar.}\\ \text{between assets}\end{pmatrix} \\
&\quad= \left(1-\frac{1}{N}\right)\overline{\text{cov}}
\end{aligned}
$$

Decomposing portfolio risk

Putting these two terms together yields

$$\sigma_p^2 = \frac{1}{N}\overline{\sigma}^2 + \left(1 - \frac{1}{N}\right)\overline{\text{cov}}$$

As $N \to \infty$, this becomes

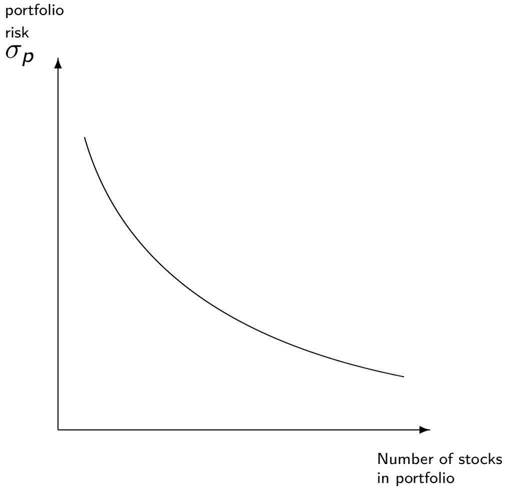
$$\sigma_p = \overline{\text{cov}}$$

The variance of a well-diversified portfolio is equal to the (weighted) average covariance.
Question: What is the variance of a portfolio composed of assets that are all uncorrelated?

Empirically: The link between number of stocks in a portfolio and portfolio volatility

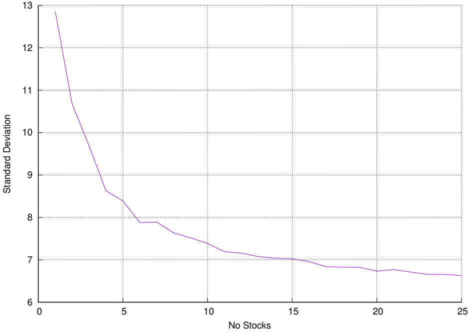
How the risk (measured by standard deviation) of a portfolio changes as the number of securities in the portfolio changes.

Typical picture:

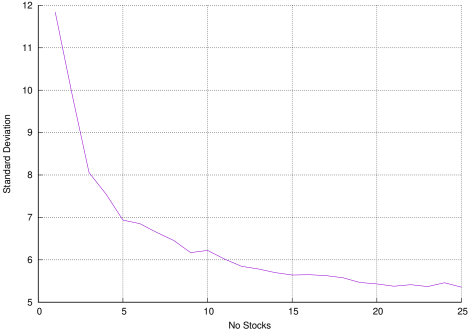


Illustrate with actual numbers from Norway.
Start with a portfolio of 1 randomly picked stock, calculate its portfolio variance, simulate 100 times, calculate the average portfolio variance.
Go on to do the same calculations for 2, 3, .. randomly picked stocks.
Do this with data for the Oslo Stock Exchange, starting in 1980

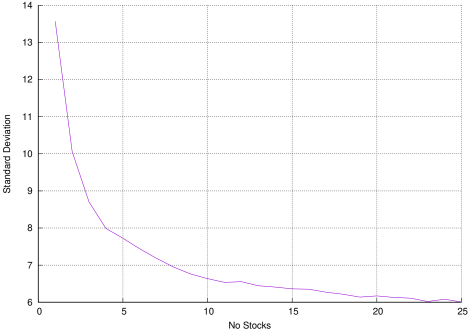
Period 1980-1999

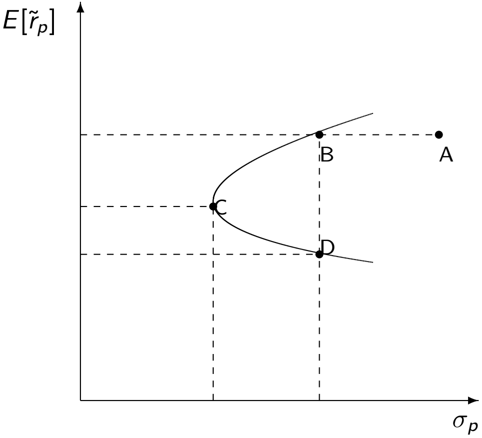


Period 2000-2020



Period 1980-2020





$E[\tilde{r}_p]$
B
A
C
D
σ_p

The *minimum variance set* is the set of portfolios that provide the lowest variance (or standard deviation) for a given level of expected return.

The *efficient set* is the set of portfolios that provide the highest expected return for a given level of variance (standard deviation).

The *global minimum variance portfolio* is the portfolio with the lowest variance among all possible portfolios.

Question: Which portfolios will investors be willing to hold?

Introducing a riskless asset

Suppose that in addition to the set of risky assets, you have the opportunity to invest in a *riskless* asset (such as a US Treasury Bill) yielding a return of r_f. A riskless asset has both a zero variance and a zero covariance with every other security.
The expected return and variance of a portfolio consisting of a fraction ω of the riskless asset and $(1-\omega)$ of a risky asset (or portfolio) are:

$$E[\tilde{r}_p] = \omega r_f + (1-\omega)E[\tilde{r}_j]$$

and

$$\sigma_p^2 = (1-\omega)^2\sigma_j^2 \qquad \text{Variance}$$

$$\sigma_p = (1-\omega)\sigma_j \qquad \text{SD}$$

The set of possible portfolios consisting of the riskless asset and some risky asset (or portfolio) j