Mean Variance Analysis

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Contents

1 Introduction **2**

2 Portfolio opportunities **2**

3 Basic tools: Expectation, Variance, Covariance. **2**
3.1 Expectation . . . 2
3.2 Variance . . . 3
3.3 Standard Deviation. . . . 3
3.4 Covariance. . . . 4
3.5 The correlation. . . . 4

4 The mean variance paradigm for quantifying risk **5**
4.1 Measuring Portfolio returns . . . 5
4.2 Measuring portfolio risk . . . 5
4.2.1 Two asset case . . . 5
4.2.2 General Case . . . 6

5 Decomposing portfolio risk **7**

6 The link between number of stocks in a portfolio and portfolio volatility **8**
6.1 The case of the US . . . 8

7 The minimum variance set. **9**

8 Introducing a riskless asset. **9**

9 Summary – mean variance **10**
9.1 Statistical tools . . . 10
9.2 Mean Variance Analysis . . . 11

10 Notation **11**

1 Introduction

Lecture overview:

- Why only care about means and variances?
- The mechanics of mean-variance calculations
- Implications for feasible portfolios
  - - minimum variance set
  - - efficient set
- Equilibrium consequence: CAPM
- Using the CAPM

2 Portfolio opportunities

Working towards a way of quantifying risk, the first step to price it.
Will do so in the context of a *portfolio* problem.
Personal portfolio selection
Tradeoff expected return and risk

The starting point for formulating this decision problem is to simplify it into a tradeoff between

- Increase in value of a portfolio
  - Measured by expected return
- The variability of return
  - Measured by standard deviation of returns

Boils down to assuming investors have preferences defined over means ($E[r]$) and variances $\sigma^2(r)$.

$$U(p) = U(E[r_p], \sigma^2(r_p))$$

Want more returns

$$\frac{\ell U}{\ell E[r]} > 0$$

Dislike risk

$$\frac{\ell U}{\ell \sigma^2(r)} < 0$$

To formalize, essentially follows straightforward from calculation of means and variances of portfolios. We therefore start there, with that calculation.

3 Basic tools: Expectation, Variance, Covariance.

3.1 Expectation

Generally, given a set of possible states (outcomes), the expected value is the sum of probabilities times outcomes.

In the case where we have only two possible outcomes, $X = X_1$ or $X = X_2$, we define the expectation as

$$E[X] = P(X_1) \cdot X_1 + P(X_2) \cdot X_2$$

where $P(X_1)$ is the probability that outcome 1 will happen, and $P(X_2)$ is the probability that outcome 2 will happen.

Exercise 1.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the expected value next period for each of the three investment opportunities

3.2 Variance

The variance of a random variable X is defined as

$$\text{var}(X) = \sigma^2(X) = E\left[(X - E[X])^2\right]$$

Exercise 2.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the variance of the expected value for each of the three investment opportunities

2. Calculate the standard deviation of this value for each of the three assets.

There are some common rules you should remember.

- Variances are not, in general, additive:

$$\begin{aligned} &\text{var}(\tilde{X} + \tilde{Y}) \\ &= \quad \text{var}(\tilde{X}) + 2\text{cov}(\tilde{X}, \tilde{Y}) + \text{var}(Y) \end{aligned}$$

- The variance of a constant a times a random variable X:

$$\text{var}(a\tilde{X}) = a^2 \text{var}(\tilde{X})$$

3.3 Standard Deviation.

The variance is problematic to interpret, you can not easily get a handle on the scale of the variance relative to the expectation. We therefore often use a measure of dispersion that is easier to interpret; the standard deviation. It is defined as the square root of the variance.

$$\sigma(X) = \sqrt{\text{var}(X)}$$

3.4 Covariance.

$$\begin{aligned}\text{cov}(X,Y) &= \sigma_{X,Y} \\ &= E\left[(X-E[X])(Y-E[Y])\right]\end{aligned}$$

The *covariance* $\sigma_{X,Y}$ is a measure of the degree to which the two variables move together. Intuitively, if X and Y tend to move in the same direction, the covariance will be positive. If they move in opposite directions, the covariance will be negative.

Note that the covariance of a random variable with itself is the variance:

$$\begin{aligned}\text{cov}(X,X) &= E\left[(X-E[X])(X-E[X])\right] \\ &= E\left[(X-E[X])^2\right] \\ &= \text{var}(X)\end{aligned}$$

Before proceeding, it will be helpful to review some important properties of covariances.

- Covariances are additive.
$$\text{cov}(\tilde{X}+\tilde{Y},\tilde{Z}) = \text{cov}(\tilde{X},\tilde{Z}) + \text{cov}(\tilde{Y},\tilde{Z})$$
- Covariance of a constant with a random variable is zero.
$$\text{cov}(c,\tilde{X}) = 0$$
- Constants multiplying random variables can be factored outside the covariance.
$$\text{cov}(c\cdot\tilde{X}, b\cdot\tilde{Y}) = c\cdot b\cdot\text{cov}(\tilde{X},\tilde{Y})$$
- Putting it all together.
$$\begin{aligned}&\text{cov}(c\tilde{X}+b\tilde{Y}-d,\tilde{Z}) \\ &= c\cdot\text{cov}(\tilde{X},\tilde{Z}) + b\cdot\text{cov}(\tilde{Y},\tilde{Z})\end{aligned}$$

What is $\text{cov}(\tilde{X}+\tilde{Y},\tilde{W}+\tilde{Z})$?

3.5 The correlation.

The covariance, like the variance, is hard to relate to the mean in a meaningful way. We therefore want a relative measure of how much two variables covary. The most common relative measure is the correlation coefficient, which we define as

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

It is always the case that

$$-1 \le \rho \le 1$$

Technical terms

$\rho = 1$	Perfect positive correlation
$1 > \rho > 0$	Positive correlation
$\rho = 0$	Uncorrelated
$0 > \rho > -1$	Negative correlation
$\rho = -1$	Perfect negative correlation

Before we leave correlation, note that we can write the covariance between two variables X and Y in terms of the correlation coefficient:

$$\text{cov}(\tilde{X}, \tilde{Y}) = \sigma(\tilde{X})\sigma(\tilde{Y})\rho_{X,Y}$$

Exercise 3.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the covariance of the values next period for the two stocks
2. Calculate the correlation between the two stocks.

4 The mean variance paradigm for quantifying risk

4.1 Measuring Portfolio returns

The return on a portfolio of securities, $\tilde{r}_p$, is a weighted average of the returns on the individual securities making up the portfolio.

$$\tilde{r}_p = \sum_{j=1}^{N} \omega_j \tilde{r}_j$$

The *expected* return on the portfolio is

$$\begin{aligned} E[\tilde{r}_j] &= E\left[\sum_{j=1}^{N} \omega_j \tilde{r}_j\right] \\ &= \sum_{j=1}^{N} \omega_j E[\tilde{r}_j] \end{aligned}$$

4.2 Measuring portfolio risk

The variance of the rate of return on a portfolio, σ_p^2

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

4.2.1 Two asset case

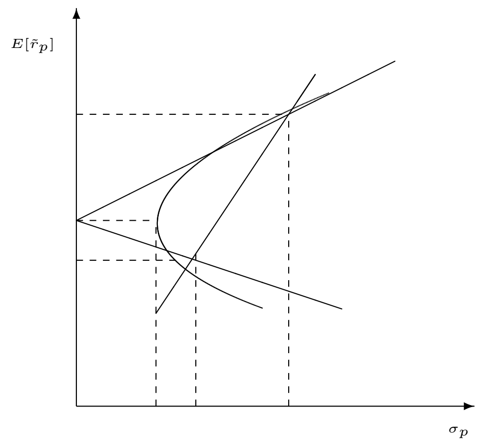
$$\begin{aligned} &\text{Variance of two-asset portfolio} \\ &= \sum_{j=1}^{2}\sum_{j=1}^{2} \omega_j \omega_i \sigma_{ij} \\ &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12} \end{aligned}$$

Exercise 4.

Suppose you hold a portfolio of two stocks. The relevant information on these two stocks is given below.

Stock	Weight	Variance	Expected return
1	0.6	0.04	0.12
2	0.4	0.09	0.20

1. Compute the expected return and variance of your portfolio assuming ρ_{12} is 0, -1 and 1.
2. Sketch how the portfolio variance and expectation would vary for these three cases.
3. Find the set of portfolio weights that minimizes the portfolio variance



Note that the portfolio weights on stock 1 in a two-stock problem that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

4.2.2 General Case

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

Substituting for $\tilde{r}_p$ and $E[\tilde{r}_p]$ yields:

$$\begin{aligned}
\sigma_p^2 &= E\left[\left(\sum_{j=1}^{N}\omega_j\tilde{r}_j - \sum_{j=1}^{N}\omega_j E[\tilde{r}_j]\right)^2\right] \\
&= E\left[\left(\sum_{j=1}^{N}\omega_j(\tilde{r}_j - E[\tilde{r}_j])\right)^2\right] \\
&= E\left[\sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])\right] \\
&= \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])]
\end{aligned}$$

But notice that these expectations are

$$\begin{aligned}
&E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])] \\
&\quad = \quad \mathrm{cov}(\tilde{r}_j, \tilde{r}_i) \\
&\quad = \quad \sigma_{ji}.
\end{aligned}$$

Therefore, we can write the variance of a portfolio as

$$\sigma_p^2 = \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_j \omega_i \sigma_{ij}$$

The correlation coefficient is computed as

$$\begin{aligned}
\rho_{ij} &= \frac{\sigma_{ij}}{\sigma_i \sigma_j} \\
&= \frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_j)}{\sqrt{\mathrm{var}(\tilde{r}_i)}\sqrt{\mathrm{var}(\tilde{r}_j)}} \\
&= \frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_j)}{SD(\tilde{r}_i)SD(\tilde{r}_j)}
\end{aligned}$$

Three alternative ways to write the variance of a portfolio.

$$\begin{aligned}
\sigma_p^2 &= \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_j \omega_i \sigma_{ij} \\
&= \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_j \omega_i \rho_{ij} \sigma_j \sigma_i \\
&= \sum_{j=1}^{N} \omega_j \sigma_{jp}
\end{aligned}$$

The contribution of security j to the variance of a portfolio is $\omega_j \sigma_{jp}$.

5 Decomposing portfolio risk

$$\begin{aligned}
\sigma_p^2 &= \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_i \omega_j \sigma_{ji} \\
&= \sum_{j=1}^{N} \omega_j^2 \sigma_j^2 + \sum_{j=1}^{N}\sum_{i \neq j}^{N} \omega_j \omega_i \sigma_{ji}
\end{aligned}$$

The variance of an equally-weighted portfolio is

$$\sigma_p^2 - \sum_{j=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_j^2 + \sum_{j=1}^{N}\sum_{i \neq j}^{N} \left(\frac{1}{N}\right)^2 \sigma_{ji}$$

The first term can be rewritten as

$$\begin{aligned}
\sum_{j=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_j^2 &= \frac{1}{N} \sum_{j=1}^{N} \frac{1}{N} \sigma_j^2 \\
&= \frac{1}{N} \overline{\sigma}^2
\end{aligned}$$

where $\overline{\sigma}^2$ is the *average variance*.

The second term can be written as

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{i\neq j}^{N}\left(\frac{1}{N}\right)^2\sigma_{ij} \\
=\ &\frac{N-1}{N^2}\sum_{j=1}^{N}\sum_{i\neq j}\frac{\sigma_{ji}}{N-1} \\
=\ &\frac{N-1}{N}\sum_{j=1}^{N}\frac{1}{N}\left(\begin{array}{l}\text{Avg. covar.}\\ \text{with asset } j\end{array}\right) \\
=\ &\frac{N-1}{N}\left(\begin{array}{l}\text{Avg. covar.}\\ \text{between assets}\end{array}\right) \\
=\ &\left(1-\frac{1}{N}\right)\overline{\text{cov}}
\end{aligned}$$

Putting these two terms together yields

$$\sigma_p^2 = \frac{1}{N}\overline{\sigma}^2 + \left(1-\frac{1}{N}\right)\overline{\text{cov}}$$

As $N \to \infty$, this becomes

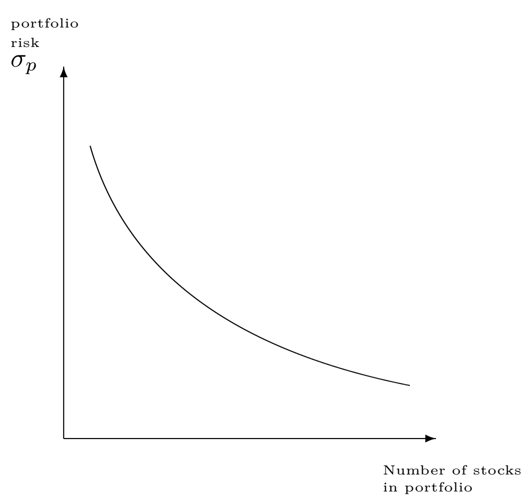
$$\sigma_p = \overline{\text{cov}}$$

The variance of a well-diversified portfolio is equal to the (weighted) average covariance.

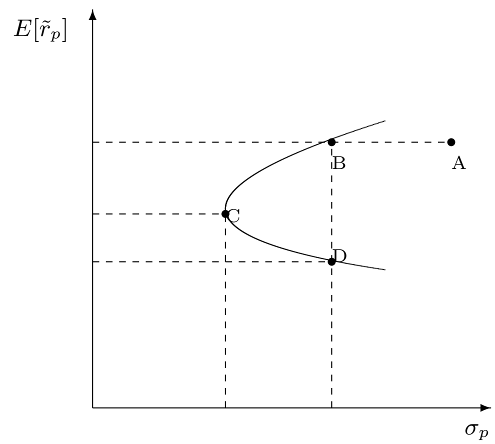
6 The link between number of stocks in a portfolio and portfolio volatility

6.1 The case of the US

How the risk of a portfolio changes as the number of securities in the portfolio changes.



7 The minimum variance set.



The *minimum variance set* is the set of portfolios that provide the lowest variance (or standard deviation) for a given level of expected return.

The *efficient set* is the set of portfolios that provide the highest expected return for a given level of variance (standard deviation).

The *global minimum variance portfolio* is the portfolio with the lowest variance among all possible portfolios.

8 Introducing a riskless asset.

Suppose that in addition to the set of risky assets, you have the opportunity to invest in a *riskless* asset (such as a US Treasury Bill) yielding a return of r_f. A riskless asset has both a zero variance and a zero covariance with every other security.

The expected return and variance of a portfolio consisting of a fraction ω of the riskless asset and $(1-\omega)$ of a risky asset (or portfolio) are:

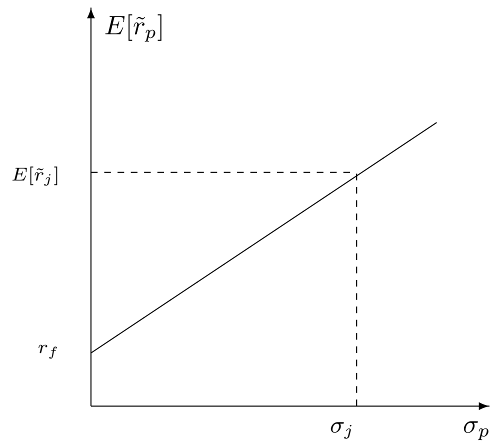
$$E[\tilde{r}_p] = \omega r_f + (1-\omega)E[\tilde{r}_j]$$

and

$$\sigma_p^2 = (1-\omega)^2\sigma_j^2 \qquad \text{Variance}$$

$$\sigma_p = (1-\omega)\sigma_j \qquad \text{SD}$$

The set of possible portfolios consisting of the riskless asset and some risky asset (or portfolio) j looks like



9 Summary – mean variance

9.1 Statistical tools

Expectation

$$E[\tilde{X}] = P(X_1) \cdot X_1 + P(X_2) \cdot X_2$$

$$\mathrm{var}(\tilde{X}) = \sigma^2(\tilde{X}) = E\left[\left(\tilde{X} - E[X]\right)^2\right]$$

Results about variance

$$\mathrm{var}(\tilde{X} + \tilde{Y}) = \mathrm{var}(\tilde{X}) + 2\mathrm{cov}(\tilde{X}, \tilde{Y}) + \mathrm{var}(\tilde{Y})$$

$$\mathrm{var}(a\tilde{X}) = a^2 \mathrm{var}(\tilde{X})$$

Standard deviation

$$\sigma(\tilde{X}) = \sqrt{\mathrm{var}(X)}$$

Covariance

$$\mathrm{cov}(\tilde{X}, \tilde{Y}) = \sigma_{X,Y} = E\left[(\tilde{X} - E[X])(\tilde{Y} - E[Y])\right]$$

Results about covariance:

$$\mathrm{cov}(\tilde{X} + \tilde{Y}, \tilde{Z}) = \mathrm{cov}(\tilde{X}, \tilde{Z}) + \mathrm{cov}(\tilde{Y}, \tilde{Z})$$

$$\mathrm{cov}(c, \tilde{X}) = 0$$

$$\mathrm{cov}(c \cdot \tilde{X}, b \cdot \tilde{Y}) = c \cdot b \cdot \mathrm{cov}(\tilde{X}, \tilde{Y})$$

Correlation

$$\rho_{X,Y} = \frac{\mathrm{cov}(X, Y)}{\sigma(X)\sigma(Y)}$$

$$\mathrm{cov}(\tilde{X}, \tilde{Y}) = \sigma(\tilde{X})\sigma(\tilde{Y})\rho_{X,Y}$$

9.2 Mean Variance Analysis

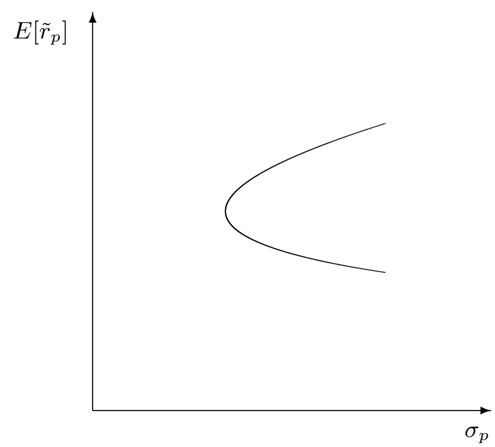
$$E[\tilde{r}_p] = E\left[\sum_{j=1}^{N}\omega_j\tilde{r}_j\right] = \sum_{j=1}^{N}\omega_j E[\tilde{r}_j]$$
$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

Variance of two-asset portfolio

$$\begin{aligned} &= \sum_{j=1}^{2}\sum_{j=1}^{2}\omega_j\omega_i\sigma_{ij} \\ &= \omega_1^2\sigma_1^2 + 2\omega_1\omega_2\sigma_{12} + \omega_2^2\sigma_2^2 \\ &= \omega_1^2\sigma_1^2 + 2\omega_1\omega_2\rho_{12}\sigma_1\sigma_2 + \omega_2^2\sigma_2^2 \end{aligned}$$

$$\sigma_p^2 = \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i\sigma_{ij}$$

Minimum variance set: Set of portfolios with the lowest variance for a given expexted return



Efficient Frontier: Set of portfolios with highest expected return for a given variance/standard deviation.

10 Notation

$\sigma(r_i)$ standard deviation of return on asset i.
$\sigma^2(r_i)$ variance of return on asset i.
$\rho(r_i, r_j)$ correlation between returns of assets i and j.
$\beta_i = \frac{mboxcov(r_i,r_m)}{\mathrm{var}(r_m)} - -$ Beta
r_D cost of debt capital
r_E cost od equity capital
D market value of debt
E market value of equity
ω_i weight of asset i in portfolio

References

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