Mean Variance Analysis

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2 Portfolio opportunities

Portfolio problem.

Investors have preferences defined over portfolio means ($E[r_p]$) and variances $\sigma^2(r_p)$.

$$U(p) = U(E[r_p], \sigma^2(r_p))$$

Want more returns

$$\frac{\ell U}{\ell E[r_p]} > 0$$

Dislike risk

$$\frac{\ell U}{\ell \sigma_p^2(r)} < 0$$

3 Basic tools: Expectation, Variance, Covariance.

3.1 Expectation

expected value: sum of probabilities times outcomes.

In the case where we have only two possible outcomes, $X = X_1$ or $X = X_2$

$$E[X] = P(X_1) \cdot X_1 + P(X_2) \cdot X_2$$

$P()$ is the probability

Exercise 1.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the expected value next period for each of the three investment opportunities

Solution to Exercise 1.

Expected values

$$\begin{aligned} E[\text{Bond}] &= \tfrac{1}{2}100 + \tfrac{1}{2}100 \\ &= 100 \\ E[\text{Stock 1}] &= \tfrac{1}{2}120 + \tfrac{1}{2}80 \\ &= 100 \\ E[\text{Stock 2}] &= \tfrac{1}{2}40 + \tfrac{1}{2}160 \\ &= 100 \end{aligned}$$

3.2 Variance

The variance of a random variable X is defined as

$$\mathrm{var}(X) = \sigma^2(X) = E\left[(X - E[X])^2\right]$$

Exercise 2.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the variance of the expected value for each of the three investment opportunities
2. Calculate the standard deviation of this value for each of the three assets.

Solution to Exercise 2.

1. Variances

$$\begin{aligned} \text{var(Bond)} &= \tfrac{1}{2}(100-100)^2 + \tfrac{1}{2}(100-100)^2 \\ &= 0 \\ \text{var(Stock 1)} &= \tfrac{1}{2}(120-100)^2 + \tfrac{1}{2}(80-100)^2 \\ &= 400 \\ \text{var(Stock 2)} &= \tfrac{1}{2}(40-100)^2 + \tfrac{1}{2}(160-100)^2 \\ &= 3600 \end{aligned}$$

2. Standard deviation

$$\begin{aligned} SD(\text{Bond}) &= \sqrt{0} &= 0 \\ SD(\text{Stock 1}) &= \sqrt{400} &= 20 \\ SD(\text{Stock 2}) &= \sqrt{3600} &= 60 \end{aligned}$$

- Variances are not, in general, additive:

$$\mathrm{var}(\tilde{X} + \tilde{Y}) = \mathrm{var}(\tilde{X}) + 2\mathrm{cov}(\tilde{X}, \tilde{Y}) + \mathrm{var}(Y)$$

- The variance of a constant a times a random variable X:

$$\mathrm{var}(a\tilde{X}) = a^2 \mathrm{var}(\tilde{X})$$

3.3 Standard Deviation.

Standard deviation is defined as the square root of the variance.

$$\sigma(X) = \sqrt{\mathrm{var}(X)}$$

3.4 Covariance.

$$\begin{aligned} \text{cov}(X,Y) &= \sigma_{X,Y} \\ &= E\left[(X - E[X])(Y - E[Y])\right] \end{aligned}$$

The *covariance* $\sigma_{X,Y}$ is a measure of the degree to which the two variables move together. The covariance of a random variable with itself is the variance:

$$\begin{aligned} \text{cov}(X,X) &= E\left[(X - E[X])(X - E[X])\right] \\ &= E\left[(X - E[X])^2\right] \\ &= \text{var}(X) \end{aligned}$$

Some important properties of covariances.

- Covariances are additive.
$$\text{cov}(\tilde{X}+\tilde{Y},\tilde{Z}) = \text{cov}(\tilde{X},\tilde{Z}) + \text{cov}(\tilde{Y},\tilde{Z})$$
- Covariance of a constant with a random variable is zero.
$$\text{cov}(c,\tilde{X}) = 0$$
- Constants multiplying random variables can be factored outside the covariance.
$$\text{cov}(c\cdot\tilde{X}, b\cdot\tilde{Y}) = c\cdot b\cdot\text{cov}(\tilde{X},\tilde{Y})$$

3.5 The correlation.

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

$$-1 \le \rho \le 1$$

$\rho = 1$	Perfect positive correlation
$1 > \rho > 0$	Positive correlation
$\rho = 0$	Uncorrelated
$0 > \rho > -1$	Negative correlation
$\rho = -1$	Perfect negative correlation

$$\text{cov}(\tilde{X},\tilde{Y}) = \sigma(\tilde{X})\sigma(\tilde{Y})\rho_{X,Y}$$

Exercise 3.

You are given the following information about three assets:

States/ Outcomes	Probability	Payoff next period		
		Bond	Stock 1	Stock 2
Recession	$\frac{1}{2}$	100	120	40
Expansion	$\frac{1}{2}$	100	80	160

1. Calculate the covariance of the values next period for the two stocks
2. Calculate the correlation between the two stocks.

Solution to Exercise 3.

1. Covariance

$$\begin{aligned} &\text{cov}(S_1, S_2) \\ &= \tfrac{1}{2}(40-100)(120-100) \\ &\quad +\tfrac{1}{2}(160-100)(80-100) \\ &= \tfrac{1}{2}(-60)(20)+\tfrac{1}{2}(60)(-20) \\ &= -1200 \end{aligned}$$

2. Correlation

$$\rho_{1,2} = \frac{-1200}{60 \cdot 20} = -1$$

   Which is perfect negative correlation.

4 The mean variance paradigm for quantifying risk

We start with the mechanics of creating mean variance optimal portfolios,

Basic idea: Construct portfolios of securities that offer the highest *expected return* for a given level of *risk*, where risk is measured by the variance/standard deviation of portfolio returns.

4.1 Measuring Portfolio returns

The return on a portfolio of securities, $\tilde{r}_p$, is a weighted average of the returns on the individual securities making up the portfolio. Therefore, if ω_j represents the proportion of the portfolio invested in security j, then

$$\tilde{r}_p = \sum_{j=1}^{N} \omega_j \tilde{r}_j$$

The portfolio weights ω_j can be either positive or negative. A *positive* weight means that you have a *long* position in the security, and a *negative* weight means that you have a *short* position in the security.

The *expected* return on the portfolio is

$$\begin{aligned} E[\tilde{r}_j] &= E\left[\sum_{j=1}^{N} \omega_j \tilde{r}_j\right] \\ &= \sum_{j=1}^{N} \omega_j E[\tilde{r}_j] \end{aligned}$$

Thus, the expected return on a portfolio is equal to the weighted average of the expected returns on the individual securities making up the portfolio.

4.2 Measuring portfolio risk.

Risk can be a difficult concept to grasp, and a great deal of controversy has surrounded attempts to define and measure it. A common definition, however, is stated in terms of the *dispersion* of possible outcomes. The tighter the probability distribution of outcomes, the smaller is the risk of the investment. The standard statistical measure of dispersion are *variance* and *standard deviation*.

The variance of the rate of return on a portfolio, σ_p^2, is computed as the expected squared deviation from the expected return on the portfolio. That is,

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

4.2.1 Two asset case

Let us look at the simplest case, where we choose between two assets.

In the special case where there are only two securities in the market, the variance of the portfolios is

$$\begin{aligned}
&\text{Variance of two-asset portfolio} \\
&= \sum_{j=1}^{2}\sum_{j=1}^{2}\omega_j\omega_i\sigma_{ij} \\
&= \omega_1^2\sigma_1^2+\omega_2^2\sigma_2^2+2\omega_1\omega_2\sigma_{12}
\end{aligned}$$

Exercise 4.

Suppose you hold a portfolio of two stocks. The relevant information on these two stocks is given below.

Stock	Weight	Variance	Expected return
1	0.6	0.04	0.12
2	0.4	0.09	0.20

1. Compute the expected return and variance of your portfolio assuming ρ_{12} is 0, -1 and 1.
2. Sketch how the portfolio variance and expectation would vary for these three cases.
3. Find the set of portfolio weights that minimizes the portfolio variance

Solution to Exercise 4.

1. The expected return on the portfolio is

$$E[\tilde{r}_p]=0.6\cdot 0.12+0.4\cdot 0.20=0.152$$

The variance of your portfolio will depend on the correlation coefficient ρ_{12}.

(a) $\rho_{12}=0$

$$\begin{aligned}
\sigma_p^2 &= \omega_1^2\sigma_1^2+\omega_2^2\sigma_2^2 \\
&= 0.6^2\cdot 0.4+0.4^2\cdot 0.09=0.0288
\end{aligned}$$

$$\sigma_p=\sqrt{0.0288}=0.1697=16.97\%$$

Note that σ_p^2 is smaller than both σ_1^2 and σ_2^2. Why?

(b) $\rho_{12}=1$

$$\begin{aligned}
\sigma_p^2 &= \omega_1^2\sigma_1^2+\omega_2^2\sigma_2^2+2\cdot\omega_1\omega_2\sigma_1\sigma_2 \\
&= (\omega_1\sigma_1+\omega_2\sigma_2)^2 \\
&= (0.6\cdot 0.2+0.4\cdot 0.3)^2 \\
&= 0.0576
\end{aligned}$$

$$\sigma_p=\sqrt{0.0576}=0.24=24\%$$

Note that when $\rho_{12}=1$, the standard deviation of the portfolio is equal to the weighted average of the standard deviations for each of the stocks in the portfolio.

$$\sigma_p=\omega_1\sigma_1+\omega_2\sigma_2$$

when $\rho_{12}=1$.

(c) $\rho_{12} = -1$

$$\begin{aligned}\sigma_p^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 - 2\cdot\omega_1\omega_2\sigma_1\sigma_2 \\ &= (\omega_1\sigma_1 - \omega_2\sigma_2)^2 \\ &= (0.6\cdot 0.2 + 0.4\cdot 0.3)^2 = 0\end{aligned}$$

$$\sigma_p = \sqrt{0} = 0\%$$

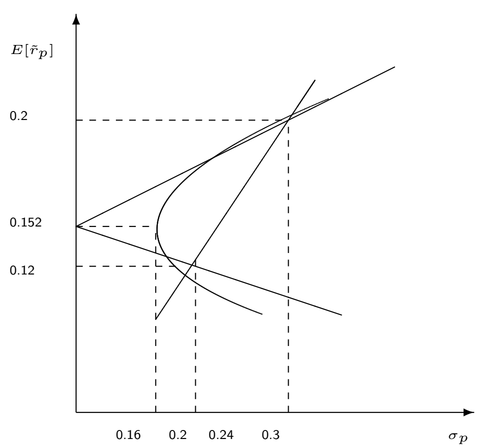
Note that when $\rho_{12} = -1$, the standard deviation of the portfolio is equal to zero if the portfolio weight on the first stock is chosen as

$$\omega_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

2. Since $\omega_1 + \omega_2 = 1$, the portfolio weight on the second stock is

$$\omega_2 = 1 - \omega_1 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

The diagram below shows the locus of portfolios you could create from stocks 1 and 2 for three different values of ρ_{12}.

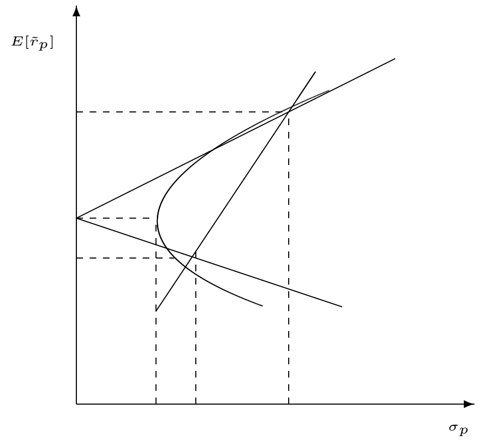


3. Note that the portfolio weights on stock 1 that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

Correlation ρ_{12}	ω_1^*	$E[\tilde{r}_p]$	σ_p
-1	0.60	15.2%	0
0	0.69	14.5%	16.64 %
1	3.00	-4.0%	0

Question: Suppose that stocks 1 and 2 are the only securities in the market. What combinations of the two stocks would investors be willing to hold in their portfolios?



Note that the portfolio weights on stock 1 in a two-stock problem that *minimises* the variance of the portfolio is

$$\omega_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

4.2.2 General Case

Let us now go back to the general case

$$\sigma_p^2 = E[(\tilde{r}_p - E[\tilde{r}_p])^2]$$

Substituting for $\tilde{r}_p$ and $E[\tilde{r}_p]$ yields:

$$\begin{aligned}
\sigma_p^2 &= E\left[\left(\sum_{j=1}^N \omega_j \tilde{r}_j - \sum_{j=1}^N \omega_j E[\tilde{r}_j]\right)^2\right] \\
&= E\left[\left(\sum_{j=1}^N \omega_j (\tilde{r}_j - E[\tilde{r}_j])\right)^2\right] \\
&= E\left[\sum_{j=1}^N \sum_{i=1}^N \omega_j \omega_i (\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])\right] \\
&= \sum_{j=1}^N \sum_{i=1}^N \omega_j \omega_i E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])]
\end{aligned}$$

But notice that these expectations are

$$\begin{aligned}
&E[(\tilde{r}_j - E[\tilde{r}_j])(\tilde{r}_i - E[\tilde{r}_i])] \\
&= \operatorname{cov}(\tilde{r}_j, \tilde{r}_i) \\
&= \sigma_{ji}.
\end{aligned}$$

Therefore, we can write the variance of a portfolio as

$$\sigma_p^2 = \sum_{j=1}^N \sum_{i=1}^N \omega_j \omega_i \sigma_{ij}$$

The *covariance* σ_{ij} is a measure of the degree to which the returns on securities i and j move together. Intuitively, if the returns on securities i and j tend to move in the same direction, the covariance will be positive. If the returns move in opposite directions, the covariance will be negative.

Another measure of the degree to which the returns on securities i and j move together is the *correlation coefficient* ρ_{ij}. The correlation coefficient is computed as

$$\begin{aligned} \rho_{ij} &= \frac{\sigma_{ij}}{\sigma_i \sigma_j} \\ &= \frac{\text{cov}(\tilde{r}_i, \tilde{r}_j)}{\sqrt{\text{var}(\tilde{r}_i)}\sqrt{\text{var}(\tilde{r}_j)}} \\ &= \frac{\text{cov}(\tilde{r}_i, \tilde{r}_j)}{SD(\tilde{r}_i)SD(\tilde{r}_j)} \end{aligned}$$

The correlation coefficient, unlike the covariance, will always take on values between -1 and 1. That is $-1 \leq \rho_{ij} \leq 1$.

The returns on securities i and j are said to be *positively* correlated if $\rho_{ij} > 0$, *negatively* correlated if $\rho_{ij} < 0$ and *uncorrelated* if $\rho_{ij} = 0$. A special case of positive correlation is when $\rho_{ij} = 1$. Then we say $\tilde{r}_i$ and $\tilde{r}_j$ are perfectly positively correlated. On the other hand if $\rho_{ij} = -1$ we say $\tilde{r}_i$ and $\tilde{r}_j$ are perfectly negatively correlated.

The correlation coefficient has another interesting interpretation. The square of the correlation coefficient is the R^2 of the regression of the returns for security i on the returns for security j. In other words, the correlation coefficient tells us something about the extent to which the returns on one security can *explain* the returns on another security.

We now have three alternative ways to write the variance of a portfolio.

$$\begin{aligned} \sigma_p^2 &= \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_j \omega_i \sigma_{ij} \\ &= \sum_{j=1}^{N}\sum_{i=1}^{N} \omega_j \omega_i \rho_{ij} \sigma_j \sigma_i \\ &= \sum_{j=1}^{N} \omega_j \sigma_{jp} \end{aligned}$$

The last line of the above equation comes from the fact that

$$\sum_{i=1}^{N} \omega_i \sigma_{ji} = \text{cov}\left(\tilde{r}_j, \sum_{i=1}^{N} \omega_i \tilde{r}_i\right) = \sigma_{jp}$$

Thus, the covariance of a portfolio is equal to the weighted average of the covariances between the return on the portfolio and the returns on the individual securities in the portfolio.

The contribution of security j to the variance of a portfolio is $\omega_j \sigma_{jp}$.

5 Decomposing portfolio risk

To better understand the role of diversification and the limits to which diversification can reduce risk, let's recall the basic formula for a portfolio consisting of N securities.

$$
\begin{aligned}
\sigma_p^2 &= \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_i\omega_j\sigma_{ji} \\
&= \sum_{j=1}^{N}\omega_j^2\sigma_j^2 + \sum_{j=1}^{N}\sum_{i\neq j}^{N}\omega_j\omega_i\sigma_{ji}
\end{aligned}
$$

The first term represent the contribution of the securities *own variance* to the variance of the portfolio. The second term represents the contribution of the *covariances* between securities to the variance of the portfolio. Note that there are N terms in the first summation (involving variances) and $N \cdot (N-1)$ terms in the second summation (involving covariances).

In the first summation, we square the weight of each term. As the number of assets increases ($N \to \infty$), the number of terms in the sum increases at the same rate as N. In a well-diversified portfolio, though, the weight on any individual asset should be getting small as N increases. When we square these small weights, they get even smaller. In fact, this provides a reasonable definition of what constitutes a well-diversified portfolio.

A *well-diversified* portfolio is one where the contribution of the own variance to the variance of the portfolio is negligible, or close to zero.

In the second summation, the number of terms is accumulating at a much faster rate, with the square of N. Thus, even though the weights in the individual asset are getting small, and begin multiplied together to make them even smaller, the number of terms is getting bigger just as fast.

To illustrate how these two summations affect the variance of a portfolio, consider a portfolio that places a constant weight $\omega_j = 1/N$ for all j, on each asset. We call this kind of portfolio an *equally weighted portfolio*. The variance of an equally-weighted portfolio is

$$
\sigma_p^2 - \sum_{j=1}^{N}\left(\frac{1}{N}\right)^2\sigma_j^2 + \sum_{j=1}^{N}\sum_{i\neq j}^{N}\left(\frac{1}{N}\right)^2\sigma_{ji}
$$

The first term can be rewritten as

$$
\begin{aligned}
\sum_{j=1}^{N}\left(\frac{1}{N}\right)^2\sigma_j^2 &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{N}\sigma_j^2 \\
&= \frac{1}{N}\overline{\sigma}^2
\end{aligned}
$$

where $\overline{\sigma}^2$ is the *average variance*.

The second term can be written as

$$
\begin{aligned}
& \sum_{j=1}^{N}\sum_{i\neq j}^{N}\left(\frac{1}{N}\right)^2\sigma_{ij} \\
&= \frac{N-1}{N^2}\sum_{j=1}^{N}\sum_{i\neq j}\frac{\sigma_{ji}}{N-1} \\
&= \frac{N-1}{N}\sum_{j=1}^{N}\frac{1}{N}\begin{pmatrix}\text{Avg. covar.}\\ \text{with asset } j\end{pmatrix} \\
&= \frac{N-1}{N}\begin{pmatrix}\text{Avg. covar.}\\ \text{between assets}\end{pmatrix} \\
&= \left(1-\frac{1}{N}\right)\overline{\text{cov}}
\end{aligned}
$$

Putting these two terms together yields

$$\sigma_p^2 = \frac{1}{N}\overline{\sigma}^2 + \left(1-\frac{1}{N}\right)\overline{\text{cov}}$$

As $N \to \infty$, this becomes

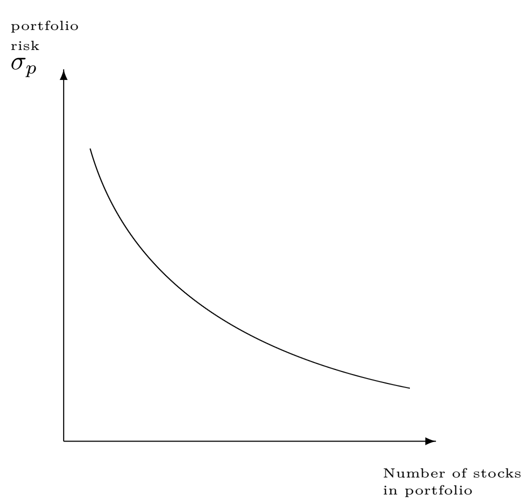
$$\sigma_p = \overline{\text{cov}}$$

The variance of a well-diversified portfolio is equal to the (weighted) average covariance.
Question: What is the variance of a portfolio composed of assets that are all uncorrelated?

6 The link between number of stocks in a portfolio and portfolio volatility

6.1 The case of the US

An interesting study was conducted by [Wagner and Lau,1971]. They looked at how the risk of a portfolio changes as the number of securities in the portfolio changes. A diagram depicting their results is provided below.



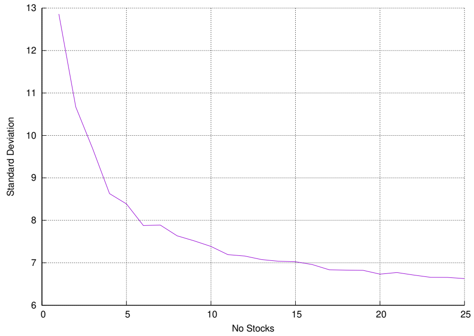
6.2 The case of Norway

Do the same kind of exercise with Norwegian data. Start with a portfolio of 1 randomly picked stock, calculate its portfolio variance, simulate 100 times, calculate the average portfolio variance.

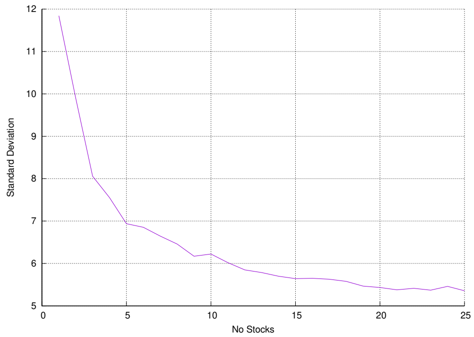
Go on to do the same calculations for 2, 3, .. randomly picked stocks. If stocks go off the exchange, replace with randomly drawn stocks.

Do this with data for the Oslo Stock Exchange, starting in 1980

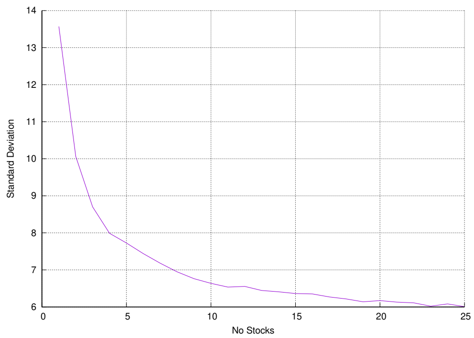
Period 1980-1999



Period 2000-2020

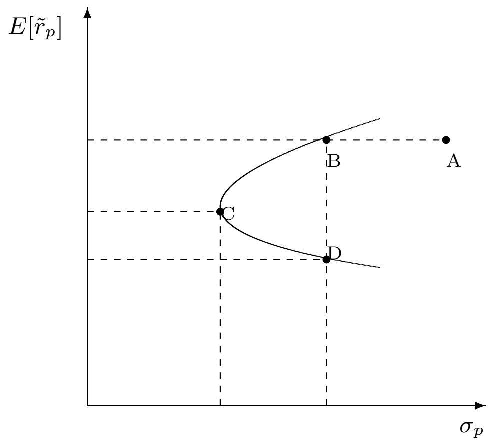


Period 1980-2020



7 The minimum variance set.

Suppose we could measure the expected returns, variances and covariances of every asset in the market. In principle, we could then compute the expected returns and variances for every portfolio that an investor could form from these assets. What would the set of these portfolios look like? The figure below show the answer to this question.



Note that the *portfolio opportunity set* look like the tip of a bullet, with the interior of the bullet included.

The *minimum variance set* is the set of portfolios that provide the lowest variance (or standard deviation) for a given level of expected return. In the picture, the minimum-variance set is represented by the perimeter of the bullet. Portfolios B, C and D fall on the minimum-variance set.

The *efficient set* is the set of portfolios that provide the highest expected return for a given level of variance (standard deviation). In the picture, the efficient set is represented bye the upward sloping section of the minimum-variance set. Portfolios B and C fall on the efficient set.

The *global minimum variance portfolio* is the portfolio with the lowest variance among all possible portfolios. In the picture, the global minimum variance portfolio is represented by the portfolio C.

Question: Which portfolios will investors be willing to hold?

8 Introducing a riskless asset.

Suppose that in addition to the set of risky assets, you have the opportunity to invest in a *riskless* asset (such as a US Treasury Bill) yielding a return of r_f. A riskless asset has both a zero variance and a zero covariance with every other security.

The expected return and variance of a portfolio consisting of a fraction ω of the riskless asset and $(1-\omega)$ of a risky asset (or portfolio) are:

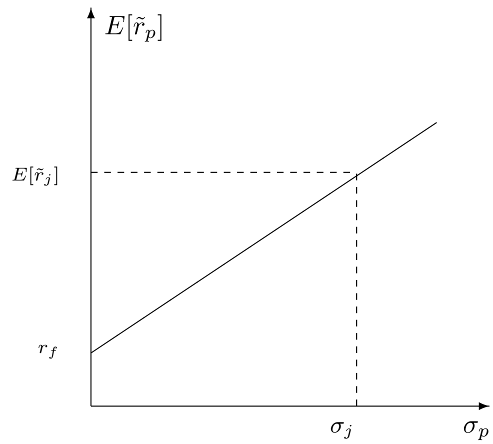
$$E[\tilde{r}_p] = \omega r_f + (1-\omega)E[\tilde{r}_j]$$

and

$$\sigma_p^2 = (1-\omega)^2\sigma_j^2 \qquad \text{Variance}$$

$$\sigma_p = (1-\omega)\sigma_j \qquad \text{SD}$$

The set of possible portfolios consisting of the riskless asset and some risky asset (or portfolio) j looks like the figure below.



Points between r_f and j represent portfolios in which there is a positive holding of both assets. Points on the line to the right of j represent portfolios in which there is a short position in the riskless asset (i.e. borrowing) with the proceeds used to purchase more of asset j.