

## Discounting - Tool number one



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Lecture topic: How to evaluate the value of cash flows arriving at different times.

Methods covered:

- ▶ (Net) Present Value (N)PV
- ▶ Internal Rate of Return IRR

Useful special cases (Mechanics of interest rate calculations)

- ▶ Perpetuities
- ▶ Annuities
- ▶ Perpetual
- ▶ Compounding
- ▶ Compounding frequency

## Relationship between Present value and Future value

*Future value*  $t$  periods from now,  $FV_t$ , of a current investment of  $PV$  dollars is

$$FV_t = PV(1 + r)^t$$

This also determines the *present value* of a future cash payment, that is:

$$PV = \frac{FV_t}{(1 + r)^t}$$

## Exercise

Suppose you have \$1000 and an opportunity to invest it and earn 10% per year for certain.

1. If you invest \$1000, how much will you have after one year?
2. Suppose after 1 year you reinvest the \$1100 at the same 10% rate of interest. How much will you have after the second year?

## Exercise solution

1. After one year

$$FV_1 = PV(1 + r) = \$1000 \cdot 1.10 = \$1100$$

2. After two years

$$FV_2 = FV_1(1 + r) = PV(1 + r)^2 = \$1000 \cdot 1.10^2 = 1210$$

## Exercise

Suppose you are saving for a trip to the Bahamas in two years and will need \$2000 at that time. The rate at which you can invest is 10%.

1. How much will you need to invest today to have enough money to make your trip two years from now?

## Exercise solution

### 1. Present value

$$FV_2 = \$2000$$

$$r = 10\%$$

$$PV = \frac{\$2000}{1.10^2} = 1653$$



## Discount factor

The ratio  $d_t = \frac{1}{(1+r)^t}$  is called the  $t$ -period *discount factor*.

# Net Present value (NPV)

To calculate the present value of an investment, we discount its expected future payoff of by the rate offered on comparable risk investments.

This rate of return is called the *discount rate*, *hurdle rate*, or *opportunity cost of capital*.

The *Net Present Value* of the investment:

Found by subtracting the initial cost from the present value of the expected future cash flow.

Alternative way to remember:

- ▶ Present value: Current value of *future* payments
- ▶ **Net** Present value: Current value of *all* payments

## Internal Rate of Return (IRR)

An alternative method of evaluating the profitability of this investment is to compare its *Internal Rate of Return* with the opportunity cost of capital. The internal rate of return is defined to be the discount rate that makes the  $NPV = 0$ .

## Exercise

Suppose you have the opportunity to invest in the development of a new, high-speed personal computer that is expected to revolutionize the computer industry. The cost of the investment is \$1 million. The computer will take exactly one year to develop, at which time you plan to sell the patent to the highest bidder. You currently expect to sell the patent for \$1.25 million. Investment in securities with similar risk are currently expected to return 20%.

1. What is the value of your investment opportunity?
2. Is this a good project?
3. Evaluate the project using the IRR method.

## Exercise solution

To calculate the present value of this investment, we discount the expected future payoff of \$1.25 million by the rate offered on comparable risk investments.

This rate of return is called the *discount rate*, *hurdle rate*, or *opportunity cost of capital*. For our example, it is 20%.

$$PV = \frac{\$1 \text{ million}}{1.20} = 1,041,667.$$

The value of your investment opportunity is however not \$1,041,667, since you are required to commit \$1 million immediately.

## Exercise solution ctd

Therefore, we must compute the *Net Present Value* of the investment by subtracting off the initial cost from the present value of the expected future cash flow.

$$\begin{aligned} NPV &= PV - \text{Initial Cost} \\ &= \$1,041,667 - 1,000,000 \\ &= 41,667 \end{aligned}$$

Is this a good project? It is positive NPV, hence yes.

## Exercise solution ctd

An alternative method of evaluating the profitability of this investment is to compare its *Internal Rate of Return* with the opportunity cost of capital. The internal rate of return is defined to be the discount rate that makes the  $NPV = 0$ .

$$NPV = \frac{\$1.25 \text{ million}}{1 + IRR} - \$1 \text{ million} = 0$$

Solving gives

$$IRR = \frac{\$1.25 \text{ million}}{\$1 \text{ million}} - 1 = 25\%$$

Since the  $IRR > r$ , the investment earns more than the opportunity cost of capital.

# Decision Rules

We now have two equivalent decision rules for making capital investments.

1. **NPV rule:** Accept all projects that have positive net present values.
2. **IRR rule:** Accept all projects that have an internal rate of return in excess of the opportunity cost of capital.

*Note:* We will later return to the IRR rule and show that it is not always equivalent to the NPV rule.



# The mechanics of discounting

Present values are additive.

The present value of a stream of cash flows is equal to the sum of the present values of each of the individual cash flows.

$$PV = PV(C_1) + PV(C_2) + \cdots + PV(C_T)$$

This formula is often referred to as the *Discounted Cash Flow (DCF)* formula.

$$PV = \sum_{t=1}^T \frac{C_t}{(1 + r_t)^t}$$

where

- ▶  $C_t$  = Cash flow in period  $t$ .
- ▶  $r_t$  = Opportunity cost of capital for cash flows occurring in period  $t$ .

To find the *Net Present value (NPV)* of a long-lived asset:  
Subtract the initial cost of the asset from its *Present Value*.

$$NPV = PV - C_0 = \sum_{t=1}^T \frac{C_t}{(1 + r_t)^t} - C_0$$

*Note:* The relationship between the term structure of interest rates and the maturity of the cash flows is called the *Term structure of interest rates*. For now we assume that  $r_t = r$  for all  $t$ , which we later will call a *flat* term structure.

## Exercise

An investment project offers the following pattern of cash flows.

Time ( $t$ )	Cash Flow ( $C_T$ )
0	-\$1000
1	500
2	750
3	250

The appropriate discount rate is 10%.

What is the NPV of the investment project?

## Exercise solution

$$\begin{aligned} PV &= \frac{500}{1.10} + \frac{750}{(1.10)^2} + \frac{250}{(1.10)^3} \\ &= 455 + 620 + 188 = 1263 \end{aligned}$$

$$\begin{aligned} NPV &= PV - C_0 \\ &= 1263 - 1000 = \underline{263} \end{aligned}$$

## Constant Perpetuities

A *perpetuity* is a security that never matures, but offers a fixed income for each year for the indefinite future. The value of such a cash flow stream is

$$PV = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

In the case where the future cash flows are the same each year,  $C_t = C$  for all  $t$ , the above formula simplifies to

$$PV = \frac{C}{r}$$

## Exercise

A *perpetuity* is a stream of cash flows which pays off a fixed amount  $C$  per period. The opportunity cost of capital is  $r$ .

1. Show that the present value of the perpetuity can be calculated as

$$PV = C \cdot \frac{1}{r}$$

## Exercise solution

$$PV = C \left[ \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots \right]$$

Can be found from formulas of infinite sums, but a cute little trick:  
Ask: What is the value of getting  $C$  in one period only?

$$\frac{C}{1+r}$$

But that is also the difference between two perpetuities where you receive  $C$  each period, one starting today, and one starting one period from now.

$$\frac{C}{1+r} = PV - \frac{PV}{1+r}$$

$$\Rightarrow PV \cdot \left( 1 - \frac{1}{1+r} \right) = C \cdot \frac{1}{1+r}$$

$$\Rightarrow PV \cdot \frac{r}{1+r} = C \cdot \frac{1}{1+r}$$

$$\Rightarrow PV = C \cdot \frac{1}{r}$$

## Exercise

Illinois Power has issued a preferred stock that pays an annual dividend of \$2.35. This dividend is fixed for the indefinite future. The opportunity cost of capital is 8%.

1. What is the market value of the preferred stock?



## Exercise solution

Market value is the present value

$$PV = \frac{2.35}{0.08} = \$29.375$$

## Growing Perpetuities

A *growing perpetuity* is a security that never matures, but offers a fixed income for each year that grows at a constant rate  $g$ . The cash flow in year  $t$  is

$$C_t = C_1(1 + g)^{t-1}$$

Substituting this into our basic perpetuity formula yields:

$$PV = \sum_{t=1}^{\infty} \frac{C_1(1 + g)^{t-1}}{(1 + r)^t}$$

After some algebraic manipulations, this formula simplifies to

$$PV = \frac{C_1}{r - g}$$

## Exercise

A stock has just paid an annual dividend of \$1.50. The dividend payments are expected to grow at a rate of 5% per year. The opportunity cost of capital is 20%.

1. What is the market value of this stock?

## Exercise solution

Market value

$$C_1 = \$1.50 \cdot 1.05 = \$1.575$$

$$PV = \frac{1.575}{0.20 - 0.05} = \$10.50$$

# Annuities

An *annuity* is an asset that pays a fixed (constant) amount each year for a specified finite number of years.

The present value of the payments from an annuity that last, say,  $T$  years is

$$\begin{aligned} PV &= \sum_{t=1}^T \frac{C}{(1+r)^t} \\ &= C \left[ \frac{1}{r} - \frac{1}{r(1+r)^T} \right] \end{aligned}$$

Values of the term in brackets is called the *annuity factor*

## Exercise

An *annuity* is an asset that pays a fixed (constant) amount  $C$  each year for a specified finite number of years.

- Show that the present value of the payments from an annuity that last  $T$  years is

$$PV = C \left[ \frac{1}{r} - \frac{1}{r(1+r)^T} \right]$$

## Exercise solution

We find the annuity formula the following way:

We know that an annuity pays  $C$  per period for  $T$  period.

Compare that to a perpetuity that pays  $C$  per period indefinitely.

Time	1	...	$T$	$T + 1$	...	Value
Perpetuity from 1 onwards	$C$	...	$C$	$C$	...	$= C \cdot \left[\frac{1}{r}\right]$
–Perpetuity from $T+1$ onwards	0	...	0	$C$	...	$= C \cdot \left[\frac{1}{(1+r)^T} \frac{1}{r}\right]$
Annuity	$C$	...	$C$	0	...	$= C \cdot \left[\frac{1}{r} - \frac{1}{(1+r)^T} \frac{1}{r}\right]$

## Exercise

You have just won the state lottery and are given the choice of receiving \$1 million immediately or \$100,000 per year for the next 20 years. The first of these payments is to be paid immediately, with subsequent payments made at one year intervals. The opportunity cost of capital is  $r = 8\%$ .

1. Which payment alternative should you choose?



## Exercise Solution

Need to find PV of the second alternative and compare to 1 million Cashflow

$$\begin{array}{rcccccc} t = & 0 & 1 & 2 & \dots & 18 & 19 \\ C_t = & 100 & 100 & 100 & \dots & 100 & 100 \end{array}$$

## Exercise Solution ctd

$$\begin{aligned}PV &= 100,000 + \sum_{t=1}^{19} \frac{100,000}{(1.08)^t} \\&= 100,000 \\&\quad + 100,000 \left[ \frac{1}{0.08} - \frac{1}{0.08 \cdot 1.08^{19}} \right] \\&= 100,000 + 100,000 \cdot 9.6036 \\&= 1,060,360\end{aligned}$$

Take the alternative that pays you \$100,000 in 20 equal annual installments.

## Compound interest

*Compounding* refers to the frequency with which interest is paid. The more frequent the compounding of interest, the higher will be the future value of a current investment of \$1 and the lower will be the present value of a future payment of \$1.

These values are computed according to the following formula:

$$FV_t = PV \left( 1 + \frac{r}{m} \right)^{mt}$$

where

- ▶  $r$  = Annual interest rate.
- ▶  $m$  = number of compounding intervals per year.

## Compound interest ctd

In the case of continuous compounding ( $m = \infty$ ) the above formula collapses to

$$FV_t = PV(e^{rt})$$

These formulas can also be used to determine the *present value* of a future cash flow:

$$PV = FV_t \left(1 + \frac{r}{m}\right)^{-mt}$$

In the case of continuous discounting, we have

$$PV = FV_t(e^{-rt})$$

## Exercise

You are asked to construct a table showing how a current investment of \$1 will grow over time as a function of the compounding interval. The interest rate is 10%.

- ▶ Show the FV after 1, 5, 10, 20 and 50 years,
- ▶ Use compounding intervals of annual, biannual, monthly, daily, as well as continuous compounding.

## Exercise solution

Compounding interval					
Year	Year	Biannual	Month	Day	Continuous
0	1.00	1.00	1.00	1.00	1.00
1	1.1000	1.1025	1.1047	1.1051	1.1052
2	1.2100	1.2155	1.2204	1.2214	1.2214
5	1.6105	1.6289	1.6453	1.6486	1.6487
10	2.5937	2.6533	2.7070	2.7177	2.7183
20	6.7275	7.0400	7.3281	7.3861	7.3891
50	117.3909	131.5013	145.3699	148.2649	148.4132

## Exercise

Your bank offers to lend you \$1000 at an annual interest rate of 12% compounded monthly. Your rich uncle offers to lend you \$1000 at an annual interest rate of  $12\frac{1}{2}\%$  compounded annually.

1. Which is the better deal?

## Exercise solution

To answer this question, we need to determine the equivalent *annually compounded* interest rate for the bank loan.

This rate is often called the *annual percentage rate*.

$$APR = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 12.6825\%$$

The loan from your uncle at an annually compounded rate of  $12\frac{1}{2}\%$  is a better deal.



## Exercise

You want to borrow \$100,000 from your bank to finance the purchase of a new home. The bank has expressed a willingness to lend you the money at an interest rate of 10% compounded monthly.

1. Determine your monthly mortgage payments if you take out a 30-year mortgage.
2. What would your monthly payments be if you take out a 15-year mortgage at the same interest rate?

## Exercise solution

30 year mortgage

Basic annuity formula:

$$PV = C \left[ \frac{1}{r} - \frac{1}{r(1+r)^T} \right]$$

This formula applies to monthly payments  $C$  if  $r$  is the appropriate monthly interest rate and  $T$  is the number of monthly payments in the annuity.

Appropriate monthly interest rate:

$$r = \frac{0.10}{12} = 0.833\%$$

On a 30-year mortgage, the number of monthly payments:

$$T = 12 \cdot 30 = 360$$

The amount borrowed is  $PV = \$100,000$ . Therefore,

$$\begin{aligned} 100,000 &= C \left[ \frac{1}{0.0083} - \frac{1}{0.0083 \cdot (1.0083)^{360}} \right] \\ &= C(120 - 6.05) = C \cdot 113.95 \end{aligned}$$

## Exercise solution ctd

15-year mortgage.

On a 15-year mortgage, the number of monthly payments equals:

$$T = 12 \cdot 15 = 180$$

The amount borrowed and the interest rate remain the same.

Therefore

$$\begin{aligned} \$100,000 &= C \left[ \frac{1}{0.0083} - \frac{1}{0.0083 \cdot (1.0083)^{180}} \right] \\ \rightarrow C &= 1,074.60 \end{aligned}$$

## Exercise

Let  $r$  be continuous compounding, and  $r_n$  be discrete compounding, where we compound  $n$  periods per year.

1. Develop general formulas to convert from either one to the other

## Exercise solution

Consider

$$e^r = \left(1 + \frac{r_n}{n}\right)^n$$

This is the future value after one year of investing one now.

If the interest rates match this equality hold

Use this to find the desired formulas

Let us first find how we go from the discrete interest rate to the continuous. Solve for  $r$ .

$$e^r = \left(1 + \frac{r_n}{n}\right)^n$$

$$\rightarrow \ln(e^r) = \ln \left\{ \left(1 + \frac{r_n}{n}\right)^n \right\}$$

$$\rightarrow r = n \ln \left(1 + \frac{r_n}{n}\right)$$

$$r = n \ln \left(1 + \frac{r_n}{n}\right)$$

## Exercise solution ctd

To go the other direction, we want to solve for  $r_n$  in

$$e^r = \left(1 + \frac{r_n}{n}\right)^n$$

$$(e^r)^{\frac{1}{n}} = \left(1 + \frac{r_n}{n}\right)^{n^{\frac{1}{n}}}$$

$$\rightarrow e^{\frac{r}{n}} = 1 + \frac{r_n}{n}$$

$$\rightarrow e^{\frac{r}{n}} - 1 = \frac{r_n}{n}$$

$$\rightarrow n \left( e^{\frac{r}{n}} - 1 \right) = r_n$$

$$\boxed{r_n = n \left( e^{\frac{r}{n}} - 1 \right)}$$

## Exercise

The interest rate with semiannual compounding is  $r_n = 10\%$ .

1. Determine the equivalent interest rate with continuous compounding.

## Exercise Solution

This corresponds to a continuous compounding of

$$\begin{aligned} r &= n \ln \left( 1 + \frac{r_n}{n} \right) \\ &= 2 \ln \left( 1 + \frac{0.1}{2} \right) = 9.78\% \end{aligned}$$



## Exercise

Interest is 8% with continuous compounding.

1. Find the equivalent quarterly interest rate.

## Exercise solution

$$\begin{aligned} r_n &= 4 \left( e^{\frac{0.08}{4}} - 1 \right) \\ &= 4 \left( e^{0.02} - 1 \right) = 8.08\% \end{aligned}$$

## Exercise

It is today 29 jun 2012, and you observe the following prices and interest rates for Norwegian State T-bills, traded at the Oslo Stock Exchange.

Ticker	Price (last)	Final Date	Coupon
NST16	99.68	19 sep 2012	-
NST17	99.29	19 dec 2012	-
NST18	99.00	20 mar 2013	-
NST18	98.62	19 jun 2013	-

On 29 jun 2012 the price of a Treasury bill maturing on 19 jun 2013 is 98.62.

What is the implicit annualised interest rate in this price?

## Exercise Solution

If we approximate the time period for this security to one year, the annual interest rate  $r$  implicit in the current bond price  $B_0$  is (remember the Treasury bill pays of 100 at maturity.)

$$B_0 = 98.62 = \frac{100}{1 + r}$$

$$r = \frac{100}{98.62} - 1 = 1.4\%$$

## Exercise Solution ctd.

However, the correct way:

The bond price corresponds to interest not over a full year, but over a period starting 29 jun 2012 and ending 19 jun 2013, which is not a full year, but 355 days.

First find the implicit continuously compounded interest rate of a period of  $\frac{355}{365} = 0.9726$  years

$$B_0 = 98.62 = e^{-\frac{355}{365}r} 100$$

Solving for  $r$ :

$$\frac{98.62}{100} = e^{-\frac{355}{365}r}$$

$$\ln(98.62) - \ln(100) = -\frac{355}{365}r$$

$$-0.01389610519 (- \cdot 365355) = r = 0.01428 \approx 1.43\%$$

## Exercise Solution ctd.

Continuously compounded interest:

$$r = 0.01428 \approx 1.43\%$$

Translating this to annualized interest  $r_n$ :

$$(1 + r_n) = e^r$$

$$r_n = e^r - 1 = 0.01439009958 \approx 1.44\%$$

The annualized interest rate implicit in the T-bill price is 1.44%

# Summary - Discounting

1. Method for evaluating current value of stream of future cashflows  
Present Value
2. Subtracting present cost  
Net Present Value  
or: Value added

# Summary - Discounting

Future Value

$$FV_t = PV(1 + r)^t$$

Present Value

$$PV = \frac{FV_t}{(1 + r)^t}$$

Discount factor

$$d_t = 1/(1 + r)^t$$

The internal rate of return: the discount rate that makes the  $NPV = 0$ .



## Summary - Discounting

Multiple cash flows – present value additive

$$PV = \sum_{t=1}^T \frac{C_t}{(1 + r_t)^t}$$

$$NPV = PV - C_0 = \sum_{t=1}^T \frac{C_t}{(1 + r_t)^t} - C_0$$

# Summary - Discounting

## Special cases

A *perpetuity*: fixed income for each year for the indefinite future.

$$PV = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

$$PV = \frac{C}{r}$$

A *growing perpetuity*: a fixed income for each year that grows at a constant rate  $g$ .

$$PV = \sum_{t=1}^{\infty} \frac{C_1(1+g)^{t-1}}{(1+r)^t}$$

$$PV = \frac{C_1}{r-g}$$

An *annuity*: pays a fixed (constant) amount each year for a specified finite number of years ( $T$ ).

$$PV = \sum_{t=1}^T \frac{C}{(1+r)^t}$$

## Summary - Discounting

*Compounding* refers to the frequency with which interest is paid.

*Discrete compounding*  $m$  times a year

$$FV_t = PV \left( 1 + \frac{r}{m} \right)^{mt}$$

$$PV = FV_t \left( 1 + \frac{r}{m} \right)^{-mt}$$

Continuous compounding

$$FV_t = PV(e^{rt})$$

$$PV = FV_t(e^{-rt})$$

Translating

$$r = n \ln \left( 1 + \frac{r_n}{n} \right)$$

$$r_n = n \left( e^{\frac{r}{n}} - 1 \right)$$