Discounting - Tool number one

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1 Intro

Lecture topic: How to evaluate the value of cash flows arriving at different times.
Methods covered:

- (Net) Present Value (N)PV
- Internal Rate of Return IRR

Useful special cases (Mechanics of interest rate calculations)

- Perpetuities
- Annuities

- Perpetual
- Compounding
- Compounding frequency

2 Introduction to present value

2.1 Relationship between Present value and Future value

Exercise 1.

Suppose you have \$1000 and an opportunity to invest it and earn 10% per year for certain.

1. If you invest \$1000, how much will you have after one year?
2. Suppose after 1 year you reinvest the \$1100 at the same 10% rate of interest. How much will you have after the second year?

Solution to Exercise 1.

1. After one year
$$FV_1 = PV(1+r) = \$1000 \cdot 1.10 = \$1100$$
2. After two years
$$FV_2 = FV_1(1+r) = PV(1+r)^2 = \$1000 \cdot 1.10^2 = 1210$$

In general, we can write the *future value* t periods from now, FV_t, of a current investment of PV dollars as
$$FV_t = PV(1+r)^t$$
The above formula is also useful for determining the *present value* of a future dollar payment, that is:
$$PV = \frac{FV_t}{(1+r)^t}$$

Exercise 2.

Suppose you are saving for a trip to the Bahamas in two years and will need \$2000 at that time. The rate at which you can invest is 10%.

1. How much will you need to invest today to have enough money to make your trip two years from now?

Solution to Exercise 2.

1. Present value
$$FV_2 = \$2000$$
$$r = 10\%$$
$$PV = \frac{\$2000}{1.10^2} = 1653$$

The ratio $1/(1+r)^t$ is called the t-period *discount factor*. Values of the discount factor for different combinations of t and r are given in the appendix of any finance text. Similarly, values for $(1+r)^t$ are given in appendices

2.2 Net Present value (NPV) - Internal Rate of Return (IRR)

To calculate the present value of an investment, we discount its expected future payoff of by the rate offered on comparable risk investments. This rate of return is called the *discount rate, hurdle rate,* or *opportunity cost of capital.* The *Net Present Value* of the investment is found by subtracting the initial cost from the present value of the expected future cash flow. An investment is good if it is positive NPV

An alternative method of evaluating the profitability of this investment is to compare its *Internal Rate of Return* with the opportunity cost of capital. The internal rate of return is defined to be the discount rate that makes the $NPV = 0$.

Exercise 3.

Suppose you have the opportunity to invest in the development of a new, high–speed personal computer that is expected to revolutionize the computer industry. The cost of the investment is $1 million. The computer will take exactly one year to develop, at which time you plan to sell the patent to the highest bidder. You currently expect to sell the patent for $1.25 million. Investment in securities with similar risk are currently expected to return 20%.

1. What is the value of your investment opportunity?
2. Is this a good project?
3. Evaluate the project using the IRR method.

Solution to Exercise 3.

1. To calculate the present value of this investment, we discount the expected future payoff of $1.25 million by the rate offered on comparable risk investments. This rate of return is called the *discount rate, hurdle rate,* or *opportunity cost of capital*. For our example, it is 20%.

$$PV = \frac{\text{\$1 million}}{1.20} = 1,041,667.$$

The value of your investment opportunity is however not $1,041,667, since you are required to commit $1 million immediately. Therefore, we must compute the *Net Present Value* of the investment by subtracting off the initial cost from the present value of the expected future cash flow.

$$\begin{aligned} NPV &= PV - \text{ Initial Cost} \\ &= \$1,041,667 - 1,000,000 \\ &= 41,667 \end{aligned}$$

2. Is this a good project? It is positive NPV, hence yes.
3. An alternative method of evaluating the profitability of this investment is to compare its *Internal Rate of Return* with the opportunity cost of capital. The internal rate of return is defined to be the discount rate that makes the $NPV = 0$.

$$NPV = \frac{\$1.25 \text{ million}}{1 + IRR} - \text{\$1 million} = 0$$

Solving gives

$$IRR = \frac{\$1.25 \text{ million}}{\text{\$1 million}} - 1 = 25\%$$

Since the $IRR > r$, the investment earns more than the opportunity cost of capital.

We now have two equivalent decision rules for making capital investments.

1. **NPV rule**: Accept all projects that have positive net present values.
2. **IRR rule**: Accept all projects that have an internal rate of return in excess of the opportunity cost of capital.

Note: We will later returns to the IRR rule and show that it is not always equivalent to the NPV rule.

3 Mathematics of finance - the mechanics of discounting.

3.1 Valuing long-lived assets

Recall our basic discounting formula for calculating the present value of a future cash flow occurring t periods from today.

$$PV = \frac{C_t}{(1+r)^t}$$

We now want to consider assets and securities that produce cash flows in more than one period in the future. For example, what would be the present value of a stream of cash flows extending over T future periods?

Fortunately, present values are additive. That is, the present value of a stream of cash flows is equal to the sum of the present values of each of the individual cash flows.

$$PV = PV(C_1) + PV(C_2) + \cdots + PV(C_T)$$

This formula is often referred to as the *Discounted Cash Flow (DCF)* formula. A short hand way of writing it is

$$PV = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t}$$

where

- C_t = Cash flow in period t.
- r_t = Opportunity cost of capital for cash flows occurring in period t.

To find the *Net Present value (NPV)* of a long-lived asset, we simply subtract the initial cost of the asset from its *Present Value.*

$$NPV = PV - C_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t} - C_0$$

Note: The relationship between the term structure of interest rates and the maturity of the cash flows is called the *Term structure of interest rates*. We are going to look at the term structure later, but for now we assume that $r_t = r$ for all t, which we later will call a *flat* term structure.

Exercise 4.

An investment project offers the following pattern of cash flows.

Time (t)	Cash Flow (C_T)
0	-$1000
1	500
2	750
3	250

The appropriate discount rate is 10%.

What is the NPV of the investment project?

Solution to Exercise 4.

$$\begin{aligned} PV &= \frac{500}{1.10} + \frac{750}{(1.10)^2} + \frac{250}{(1.10)^3} \\ &= 455 + 620 + 188 = 1263 \\ NPV &= PV - C_0 \\ &= 1263 - 1000 = 263 \end{aligned}$$

3.2 Valuing Perpetuities and Annuities

3.2.1 Constant Perpetuities

A *perpetuity* is a security that never matures, but offers a fixed income for each year for the indefinite future. The value of such a cash flow stream is

$$PV = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

In the case where the future cash flows are the same each year, $C_t = C$ for all t, the above formula simplifies to

$$PV = \frac{C}{r}$$

Exercise 5.

A *perpetuity* is a stream of cash flows which pays off a fixed amount C per period. The opportunity cost of capital is r.

1. Show that the present value of the perpetuity can be calculated as

$$PV = C \cdot \frac{1}{r}$$

Solution to Exercise 5.

$$PV = C\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right]$$

Can be found from formulas of infinite sums, but a cute little trick:

Ask: What is the value of getting C in one period only?

$$\frac{C}{1+r}$$

But that is also the difference between two perpetuities where you receive C each period, one starting today, and one starting one period from now.

$$\begin{aligned}
\frac{C}{1+r} &= PV - \frac{PV}{1+r} \\
\Rightarrow \quad PV \cdot \left(1 - \frac{1}{1+r}\right) &= C \cdot \frac{1}{1+r} \\
\Rightarrow \quad PV \cdot \frac{r}{1+r} &= C \cdot \frac{1}{1+r} \\
\Rightarrow \quad PV &= C \cdot \frac{1}{r}
\end{aligned}$$

Exercise 6.

Illinois Power has issued a preferred stock that pays an annual dividend of \$2.35. This dividend is fixed for the indefinite future. The opportunity cost of capital is 8%.

1. What is the market value of the preferred stock?

Solution to Exercise 6.

1. Market value is the present value

$$PV = \frac{2.35}{0.08} = \$29.375$$

A *growing perpetuity* is a security that never matures, but offers a fixed income for each year that grows at a constant rate g. The cash flow in year t is

$$C_t = C_1(1+g)^{t-1}$$

Substituting this into our basic perpetuity formula yields:

$$PV = \sum_{t=1}^{\infty} \frac{C_1(1+g)^{t-1}}{(1+r)^t}$$

After some algebraic manipulations, this formula simplifies to

$$PV = \frac{C_1}{r-g}$$

Note: The present value for a *constant* perpetuity is a special case of a growing perpetuity with $g = 0$.

Exercise 7.

A stock has just paid an annual dividend of \$1.50. The dividend payments are expected to grow at a rate of 5% per year. The opportunity cost of capital is 20%.

1. What is the market value of this stock?

Solution to Exercise 7.

1. Market value

$$C_1 = \$1.50 \cdot 1.05 = \$1.575$$
$$PV = \frac{1.575}{0.20 - 0.05} = \$10.50$$

3.2.2 Annuities.

An *annuity* is an asset that pays a fixed (constant) amount each year for a specified finite number of years.

The present value of the payments from an annuity that last, say, T years is

$$\begin{aligned} PV &= \sum_{t=1}^{T} \frac{C}{(1-r)^t} \\ &= C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right] \end{aligned}$$

Values of the term in brackets can be found in the appendix of any finance text.

Exercise 8.

An *annuity* is an asset that pays a fixed (constant) amount C each year for a specified finite number of years.

1. Show that the present value of the payments from an annuity that last T years is

$$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right]$$

Solution to Exercise 8.

1. We find the annuity formula the following way: We know that an annuity pays C per period for T period. Compare that to a perpetuity that pays C per period indefinitely.

Time	1	$\cdots$	T	$T+1$	$\cdots$		Value
Perpetuity from 1 onwards	C	$\cdots$	C	C	$\cdots$	$=$	$C \cdot [\frac{1}{r}]$
$-$Perpetuity from T+1 onwards	0	$\cdots$	0	C	$\cdots$	$=$	$C \cdot [\frac{1}{(1+r)^T}\frac{1}{r}]$
Annuity	C	$\cdots$	C	0	$\cdots$	$=$	$C \cdot \left[\frac{1}{r} - \frac{1}{(1+r)^T}\frac{1}{r}\right]$

Exercise 9.

You have just won the state lottery and are given the choice of receiving \$1 million immediately or \$100,000 per year for the next 20 years. The first of these payments is to be paid immediately, with subsequent payments made at one year intervals. The opportunity cost of capital is $r = 8\%$.

1. Which payment alternative should you choose?

Solution to Exercise 9.

1.

$t =$	0	1	2	$\cdots$	18	19
$C_t =$	100	100	100	$\cdots$	100	100

$$\begin{aligned} PV &= 100,000 + \sum_{t=1}^{19} \frac{100,000}{(1.08)^t} \\ &= 100,000 \\ &\quad +100.000 \left[\frac{1}{0.08} - \frac{1}{0.08 \cdot 1.08^{19}} \right] \\ &= 100,000 + 100,000 \cdot 9.6036 \\ &= 1,060,360 \end{aligned}$$

Take the alternative that pays you \$100,000 in 20 equal annual installments.

4 Summary – discounting

1. Method for evaluating current value of stream of future cashflows
   Present Value
2. Subtracting present cost
   Net Present Value
   or: Value added

Future Value

$$FV_t = PV(1+r)^t$$

Present Value

$$PV = \frac{FV_t}{(1+r)^t}$$

Discount factor

$$d_t = 1/(1+r)^t$$

Multiple cash flows – present value additive

$$PV = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t}$$

$$NPV = PV - C_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t} - C_0$$

The internal rate of return: the discount rate that makes the $NPV = 0$.
A *perpetuity*: fixed income for each year for the indefinite future.

$$PV = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

$$PV = \frac{C}{r}$$

A *growing perpetuity*: a fixed income for each year that grows at a constant rate g.

$$PV = \sum_{t=1}^{\infty} \frac{C_1(1+g)^{t-1}}{(1+r)^t}$$

$$PV = \frac{C_1}{r-g}$$

An *annuity*: pays a fixed (constant) amount each year for a specified finite number of years (T).

$$\begin{aligned} PV &= \sum_{t=1}^{T} \frac{C}{(1-r)^t} \\ &= C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right] \end{aligned}$$

The term $A_{r,T} = \left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right]$ is called an *annuity factor*.
Compounding refers to the frequency with which interest is paid.

Discrete compounding m times a year

$$FV_t = PV\left(1+\frac{r}{m}\right)^{mt}$$

$$PV = FV_t\left(1+\frac{r}{m}\right)^{-mt}$$

Continous compounding

$$FV_t = PV(e^{rt})$$

$$PV = FV_t(e^{-rt})$$

Translating

$$r = n\ln\left(1+\frac{r_n}{n}\right)$$

$$r_n = n\left(e^{\frac{r}{n}}-1\right)$$

5 Notation

FV future value
PV present value
t time at which cash flows occur
r interest rate used for discounting
C_t cash flow at time t
Abbreviations
PV present value
NPV net present value
IRR internal rate of return