Financial Derivatives – an overview

Bernt Arne Ødegaard

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1 Introduction

The unbelievable growth in the amount of derivatives contracts outstanding a major contribution of finance to society. It allows the hedging of risks that could before not be insured against.

In this lecture we look at the corporate use of *derivatives*.

Derivatives are defined as: Securities traded in financial markets that whose value depend on the price/-value of some observable(contractible) financial asset.

To motivate, consider a very common corporate situation.

2 Example: Hedging currency risk

Exercise 1.

You are the CFO of a US-based corporation. It is now the end of February '20, and you have just signed a contract that will pay Euros (EUR) 1 mill one year from now. If you do nothing, you receive next March (in dollars)

$$\text{EUR 1mill} \times \text{Spot exchange rate}$$

The current exchange rate S_t is 1.09. The spot exchange rate one year from now S_T is uncertain, but there are ways to do something about this uncertainty.

- You can enter a forward contract. Looking at the CME, you find the following *Euro FX Futures Quotes*

Maturity	Forward Price
Jul '20	1.0968
Sep '20	1.1005
Dec '20	1.1056
Mar '21	1.1108

  If you enter into a forward contract at these quotes, you commit yourself to exchanging the EUR 1 million at the forward rate in March '21.

- You can trade an option. At the CME you will also find quotes of FX options EUR/USD with maturity February 2021.

Option Price	Strike quote
37200	1.0900
34100	1.0950
31200	1.1000
28500	1.1050
26000	1.1100

The option price is the USD price for a option for 1 mill EUR with the indicated strike quote. The way the option works, you can choose to exchange the 1 mill EUR at the indicated strike. To enter into the option contract, you have to pay the premium up front.

1. Suppose you enter into a Mar '21 forward contract for 1 mill EUR, which guarantees that the USD amount you get for 1 mill EUR is 1 mill $\times$ 1.1108.
   (a) Suppose the spot exchange rate in March '21 is 1.05. What would your position have been without the forward contract? With the forward contract?
   (b) Suppose the spot exchange rate in March '21 is 1.15. What would your position have been without the forward contract? With the forward contract?
   (c) Is there a general relation between your position and the spot exchange rate in March '21?
2. Suppose you instead buy the option with a strike price of 1.1.
   (a) Suppose the spot exchange rate in March '21 is 1.05. What is the value of the option contract? What is your total position?
   (b) Suppose the spot exchange rate in March '21 is 1.15. What is the value of the option contract? What is your total position?
   (c) Is there a general relation between your position and the spot exchange rate in March '21?

3 Forward/futures contracts

Example

You go to the bookstore, look for a textbook. The textbook is out of stock. The clerk puts it on order for you, it will cost you $50 at delivery. Congratulations, you have just entered a forward contract.

Define forward contracts: Agreement to buy (sell) given *amounts* underlying at given *price* (forward price) and at given *time* (expiry date).

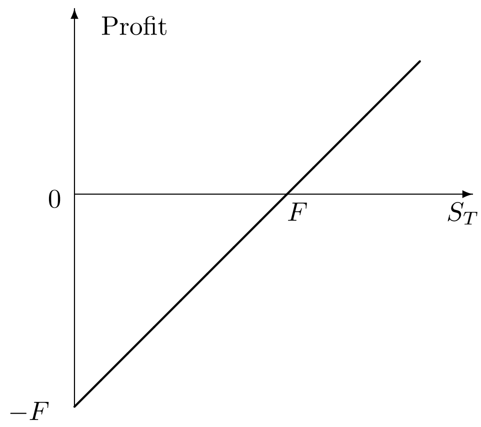
Two parties:

Long: Party buying commodity in the future (buy forward)
Short: Party selling commodity in the future (sell forward)

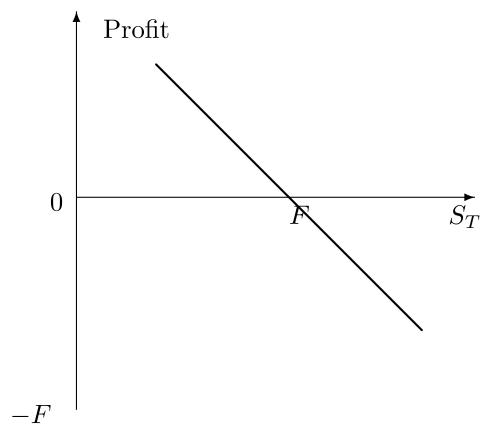
4 Forward/Futures - relation to underlying

The forward is an agreement at time t to trade at time T with a forward price F_t. What is the value of your position when the contract mature?

Payoff Long Forward



Payoff Short Forward



5 Pricing of Forward contracts

Exercise 2.

Consider a forward contract on an underlying asset that provides no income. There are also no restrictions on shortselling of the underlying asset. Let S_t and S_T denote the price of the underlying asset at t and T, respectively. r denotes the riskfree rate. Then the (time-t) forward price F_t for a contract with deliver date T has to satisfy

$$F_t = S_t(1+r)^{(T-t)},$$

i.e., the forward price is the future value of the current price of the underlying.

Use arbitrage arguments to show this.

6 Option Definitions

A *Call option* is a right to *buy* a underlying security at a fixed price (exercise price - K) in some given time period (expiry).

A *Put option* is the right to *sell* a underlying security at a fixed price (exercise price) in some given time period.

If we *use* the option to buy/sell the asset we *exercise* the option.

If the option is an *European* option, it can only be exercised at the expiry date.

If the option can be exercised any time up to the expiry date, it is called an *American* option.

7 Option payoff/profit diagrams

7.1 Payoff at maturity. Position diagrams.

Exercise 3.

Consider buying either put or call options on IBM stock with an exercise price of $X = 100$. The current price of IBM stock S_0 is 100.

1. What are the cash flows at maturity (time T) if the then stock price (S_T) is 120?
2. What are the cash flows at maturity (time T) if the then stock price (S_T) is 80?
3. Plot the payoffs from the call and put options as functions of stock price at maturity.

We can summarize the payoffs at maturity from buying options as:

- Call option: Payoff $= \max(0, S_T - X)$.
- Put option: Payoff $= \max(0, X - S_T)$.

Note that the payoff of an option is never negative.

Exercise 4.

Consider selling either put or call options on IBM stock with an exercise price of $X = 100$. The current price of IBM stock S_0 is 100.

1. What are the cash flows at maturity (time T) if the then stock price (S_T) is 120?
2. What are the cash flows at maturity (time T) if the then stock price (S_T) is 80?
3. Plot the payoffs from the call and put options as functions of stock price at maturity.

7.2 Total profits from holding options.

Exercise 5.

Put and call options on IBM stock with an exercise price of $X = 100$ are traded at \$3 and \$5, respectively. The current price of IBM stock S_0 is 100.

1. What is the total cashflow from the transaction of buying the call if the price of the underlying at maturity turns out to be 120.
2. What if the price of the underlying is 80?
3. Plot the total cash flows from buying call and put options.

Thus, to get the total profit from holding an option contract, we need to subtract the premium.

- Call option: Profit $= \max(0, S_T - X) - C_0$.
- Put option: Profit $= \max(0, X - S_T) - P_0$.

What happens if we *sell* a security.

Exercise 6.

Put and call options on IBM stock with an exercise price of $X = 100$ are traded at \$3 and \$5, respectively. The current price of IBM stock S_0 is 100.

1. Plot the total cash flows from selling (issuing) these call and put options.

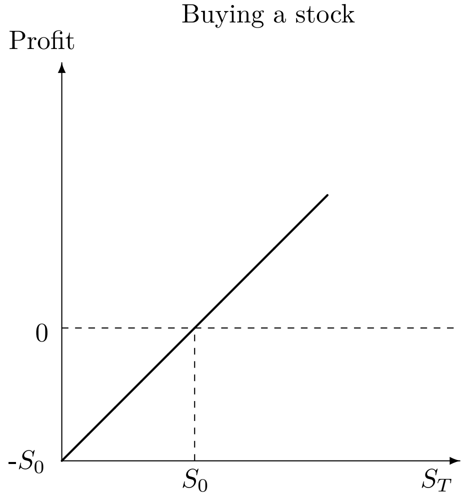
Thus, for the seller of options, the total profits can be summarized as:

- Call option: Payoff $= C_0 + \max(0, S_T - X)$.
- Put option: Payoff $= P_0 + \max(0, X - S_T)$.

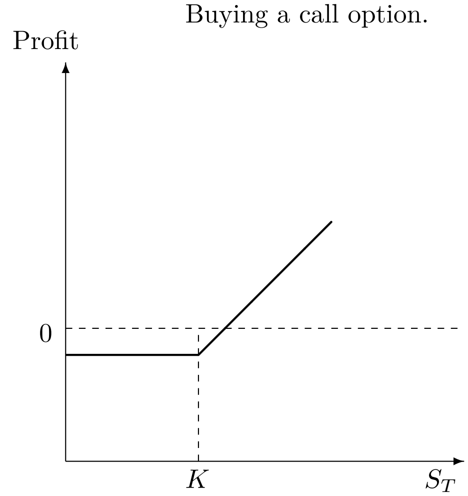
7.3 Comparing options with other securities

Profit from buying financial contracts.

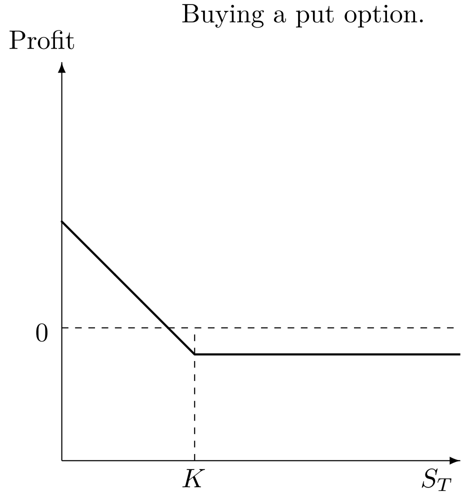
Buying a stock



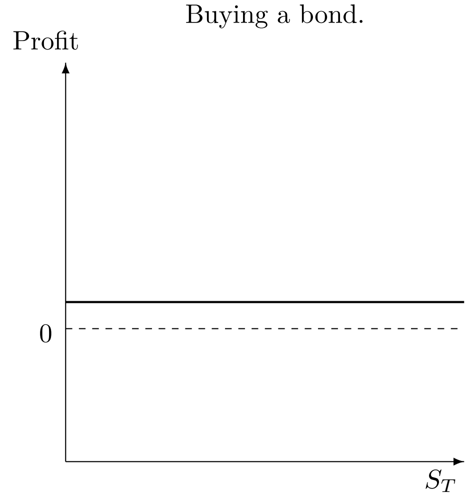
Buying a call option.



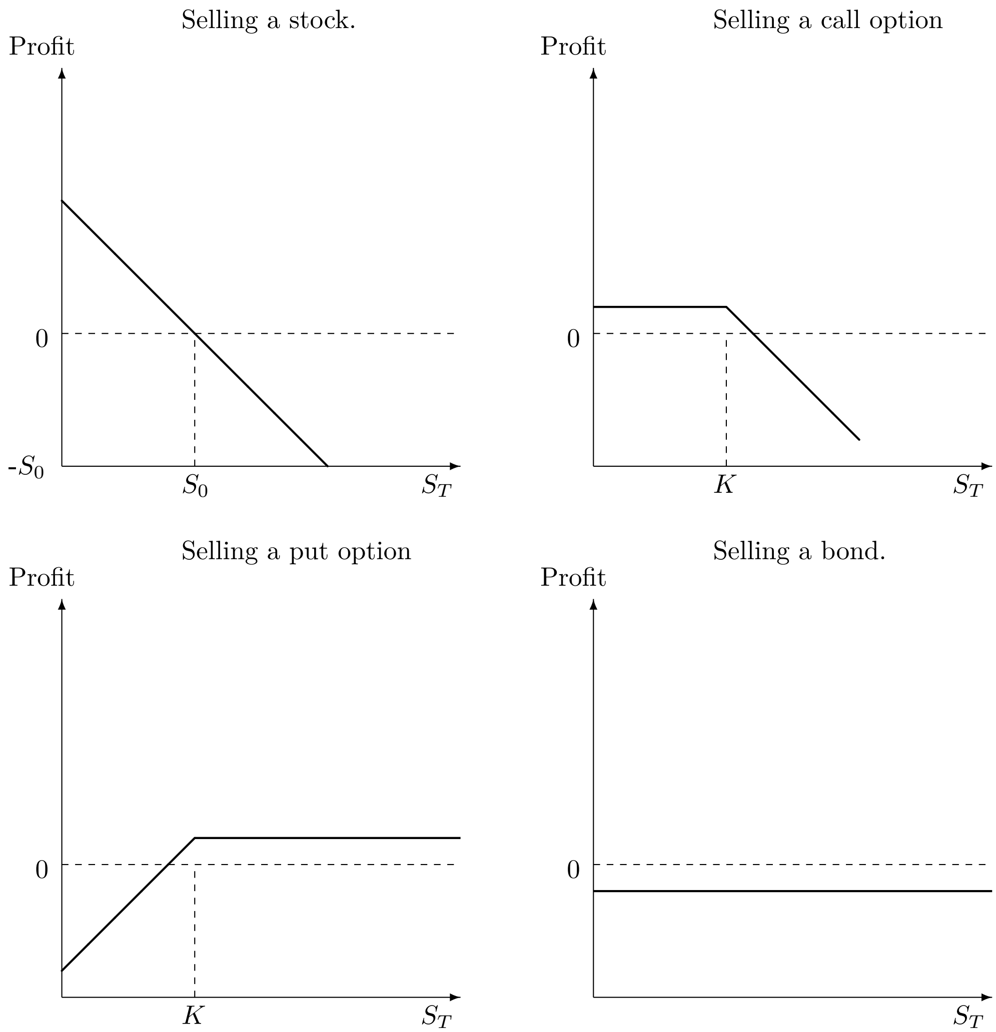
Buying a put option.



Buying a bond.



Profit from selling financial contracts.



8 On pricing of options

The ability to price contingent claims such as options and other derivative securities is a recent innovation in Finance. Indeed, the existence of a simple formula for the price of an option (The Black-Scholes formula) was one of the chief reasons for the explosive growth in these markets. The CBOE (Chicago Board of Options Exchange) started trading options on common stock in 1973, and in the same year two important papers describing option pricing formulas were published: Black and Scholes (1973) and Merton (1973)

The type of analysis used to find option values has had impact in most areas of finance. All relies on a *no-arbitrage* argument, which we can summarise as

If two portfolios or assets have the same payoffs tomorrow, they must have the same price today.

The challenge in coming up with an option pricing formula is the *contingent* feature of options, they are only used (excercised) when it results in a positive payoff.

This is different from e.g. pricing a forward. To price a forward contract, one can set up a portfolio which will replicate the payoffs from the forward using a position in the underlying security and riskfree borrowing/lending. This replicating portfolio only needs to be set up once, and it then works perfectly at the expiry of the forward.

An options contract can also be replicated using the underlying security and riskfree borrowin/lending, but the replicating portfolio needs to be changed as time passes (and the price of the underlying security changes).

Constructing the replicating strategy is a nontrivial mathematical problem.

To help in understanding how option pricing works, we do this construction in a simple setting, the *binomial* framework.

In practical use, we use a more complex algorithm, the Black Scholes formula.

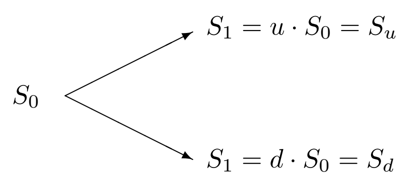
9 Stock Price movement.

One of the simplest assumption you can think of is that the stock price next period can take on only two values,

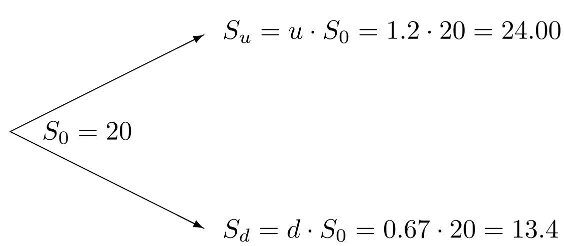
Exercise 7.

Let S_t be the stock price. It is currently equal to S_0.

Suppose we know that next period, the stock price has only two possible values. We can summarize the change from one period to the next by two multiplicative factors u and d:



$$
\begin{aligned}
S_0 &= 20 \\
u &= 1.2 \\
d &= 0.67 \\
r &= 10\%
\end{aligned}
$$



Also assume that the price jumps up with probability p and jumps down with probability $(1 - p)$.

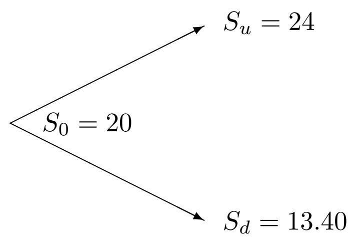
Determine the expected stock price next period with the following assumptions about the probability p:$p = \frac{1}{2}$,$p = 0.625$, and $p = \frac{2}{3}$.

The major surprise about pricing options is that the expected stock price is not relevant for the price of a call option on the stock.

9.1 Call option payoffs.

Exercise 8.

The evolution of the price S of some underlying security is given by



1. Determine the possible payoffs at time 1 of a one period call option with exercise price K

9.2 Constructing a hedge portfolio.

In order to find the call price at time 0, we now ask the following question: What combination of stock and call options will give us a riskless payoff next period? More specifically: If we buy one stock, how many call options do we need to buy/sell to make the payoff next period riskless?

Let m be the number of call options we buy. The payoff from a strategy of buying 1 stock and m call options should be the same in both the up-state and the down-state, or

$$mC_u + S_u = mC_d + S_d$$

$$\Rightarrow \quad mC_u + uS_0 = mC_d + dS_0$$

$$\Rightarrow \quad m(C_u - C_d) = (d - u)S_0$$

$$\Rightarrow \quad m = \frac{(d-u)S_0}{(C_u - C_d)}$$

Exercise 9.

The current price of the underlying security S_0 is 20. The price of the underlying follows a binomial process with up and down movements. $u = 1.2$ and $d = 0.67$. The one period risk free interest rate r equals 10%.

1. Calculate the portfolio of 1 stock and m options which has a risk free payoff next period
2. Control your calculation by calculating the portfolio payoffs as a function of the state.

Thus, the strategy of buying 1 stock and buying -2.65 calls will next period have a known payoff, and hence a certain (known) return. This return has to equal the risk free rate of return, r, to avoid arbitrage.

The cost of the portfolio is $S_0 + mC_0$.

In our example, the cost is $20 - 2.65C_0$, and the known future payment is \$13.40. The return has to equal the riskfree rate $r = 10\%$.

$$(S_0 + mC_0)e^r = 13.40$$

$$(20 - 2.65C_0)e^{0.10} = 13.40$$

$$-2.65C_0 = e^{-0.10} \cdot 13.40 - 20$$

$$C_0 = \frac{e^{-0.1} \cdot 13.40 - 20}{-2.65}$$

The price of the call is $C_0 = 2.95$.

Let us check that the return on the portfolio is equal to the riskfree rate:

$$\begin{aligned} \text{Return} &= \ln\left(\frac{13.40}{20 - 2.65 \cdot 2.95}\right) \\ &= 0.0999 \approx 10\% \end{aligned}$$

10 The Black Scholes formula

The Black Scholes formula

The Black Scholes formula for a call option is

$$c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(\cdot) = \text{ The cumulative normal distribution}$$

The price of a put option is

$$p = Ke^{-r(T-t)} N(-d_2) - SN(-d_1)$$

11 Calculating option prices using the Black Scholes formula

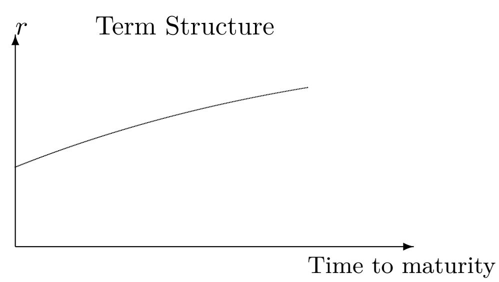
Exercise 10.

Consider 3 month options with exercise prices of $K = 45$. The variance of the underlying security is $\sigma^2 = 0.20$. The risk free interest rate is $r = 6\%$. The current price of the underlying security is $S = 30$.

1. Determine the Black Scholes prices for call and put options.
2. Check that your calculations satisfy put call parity.

12 Using the Black Scholes pricing formula.

Term structure of interest rates



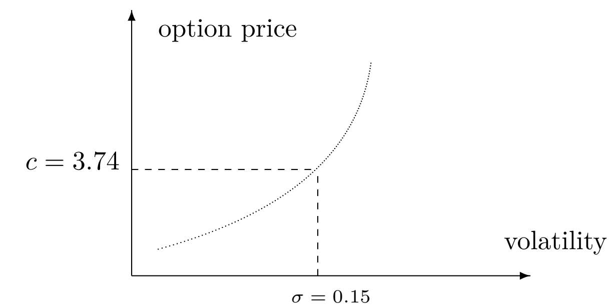
12.1 Volatility

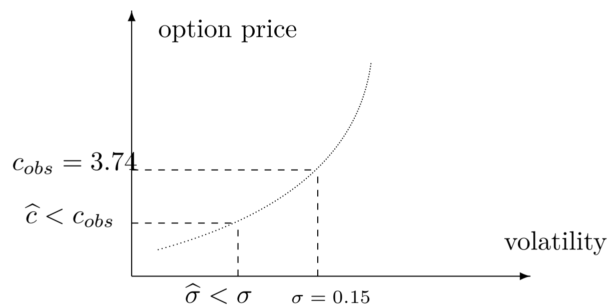
Standard deviation (σ).

Two methods for finding estimates of volatility.

12.1.1 Historical volatility.

12.1.2 Implied volatility.





13 Analyzing capital structure using option theory

We will in this section look at how we can get new insights about the choice between debt and equity by seeing how they can be interpreted as options.

13.1 Defining Debt and Equity.

Consider a firm with two classes of liabilities: Equity and Debt. Assume there is a single, homogeneous class of debt with the following terms:

1. The debt is a "pure" discount bond where the firm promises to pay M dollars for each bond at the maturity date T. If there are n bonds outstanding, then the total promised payment to the debtholders is $F = nM$ at the maturity date T.
2. In the event that the firm does not make the promised payment ("default") then the firm goes bankrupt, its assets are turned over to the bondholders, and each bondholder will receive his pro rata share of the "reorganized" firm. The original equity holders will receive nothing in that event.

Let V_t denote the market value of the firm at time t. (which, by definition, will always equal the sum of the market value of debt plus equity.)

13.2 Payoff at maturity.

On the maturity date of debt:

- If the value of the firm exceeds the amount of the promised payment, (ie $V_T > F$), then it is in the interest of the equityholders (who elect management) to have debt paid. Thus, the value of the debt issue in that event will be F, and the value of equity will be $V_T - F$.

- If the value of the firm is less than the amount of the promised payment (ie $V_T < F$), then the firm can not make the promised payment. Because corporate equity enjoys limited liability, the equity holders cannot be compelled to contribute the "short fall" to pay the bondholders, and it is, clearly, not in their interest to do so. Thus, the firm will default, and the value of the debt issue in that event will be V_T, and the value of equity will be 0.

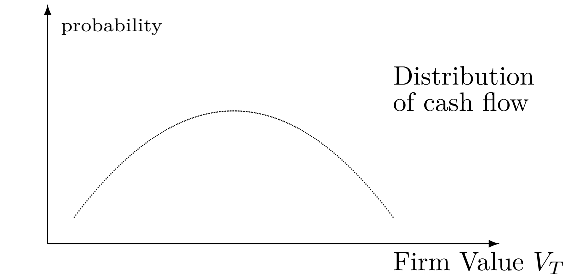
To summarize, on the maturity date:

$$\begin{aligned}\text{Value of debt } &= \min(F, V_T) \\ &= F - \max(0, F - V_T) \\ \text{Value of equity } &= V_T - \min(F, V_T) \\ &= V_T - F + \max(0, F - V_T) \\ &= \max(0, V_T - F)\end{aligned}$$

13.3 Interpreting the payoffs as options

Let us now try to interpret these payoffs in terms of options.

We have assumed that a firm has some stochastic value of V_T next period:



We can summarize the financing scheme as:

Bondholders

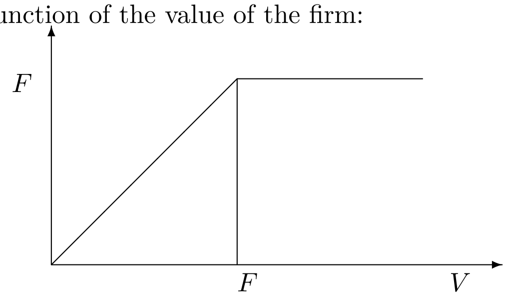
- Promised payment of F next period.
- If default occurs, bondholders own the firm.

Stockholders:

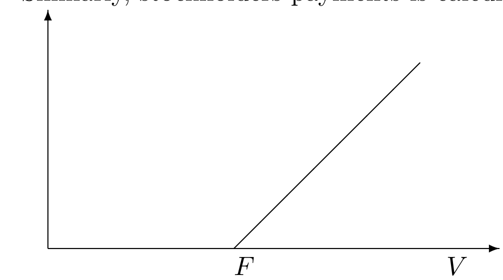
- Receive all residual cash flows after payments to bondholders.

Payments to	$V \leq F$	$V > F$
Bondholders	V	F
Stockholders	0	$V - F$
Total	V	V

We have calculated bondholders payments as $\min(V, F)$ Let us plot the payment to bondholders as a function of the value of the firm:



Similarly, stockholders payments is calculated as $\max(0, V - F)$



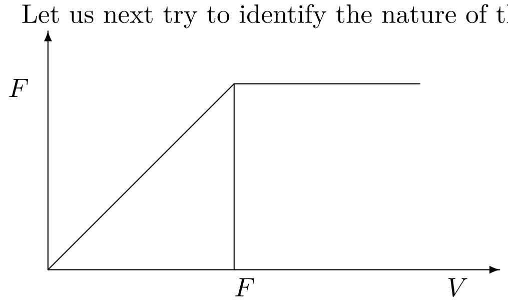
Note that payments to the stockholders and bondholders add up to the total cash flows of the firm.

$$
\begin{aligned}
&\text{Value of firm} \\
&= \min(V, F) \quad \text{(Bondholders)} \\
&\quad + \max(0, V - F) \quad \text{(Stockholders)} \\
&= \begin{cases} V + 0 = V & \text{if } V < F \\ F + (V - F) = V & \text{if } V \geq F \end{cases} \\
&= V
\end{aligned}
$$

13.4 The firm in terms of call options

Let us now see if we can identify the nature of these options. Let us start by the equity. The payoff to equity is $\max(0, V - F)$. This can be interpreted as a call option on the firm value, with exercise price F. Hence, if the firm value is less than the exercise price, do not exercise, the value of equity is zero.

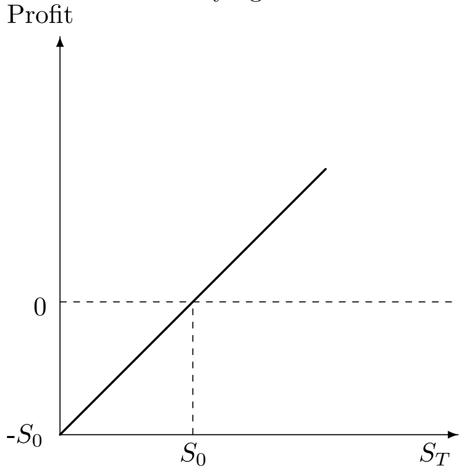
Let us next try to identify the nature of the debt.



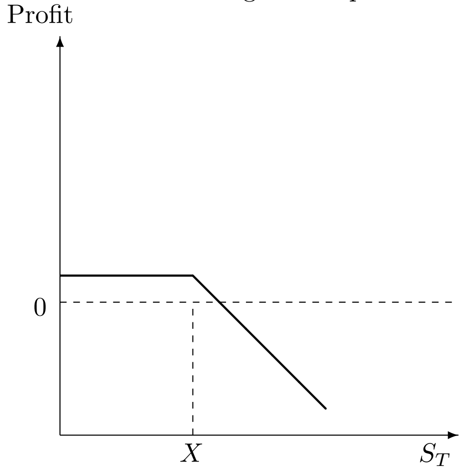
In terms of our usual stock options, we can achieve this payoff by

1. Buying the stock.
2. Selling a call.

Buying a stock.



Selling a call option.



Thus, we can view holding bonds in a firms as:

1. Owning the firm.
2. Selling a call option on the firm with exercise price F.

13.5 The firm in terms of put options

Recall the basic "put–call parity" relation which we showed earlier

$$C_0 = P_0 + S_0 + \frac{X}{1 + r_f}$$

With our interpretation of debt and equity in terms of calls, we can rewrite it in terms of puts.

We showed that equity is equal to a call with exercise price F. Using

$$C_0 = P_0 + S_0 - \frac{X}{1 + r_f}$$

By reinterpreting these in terms of the firm, we see that it is also equal to:

1. Owning the firm.
2. Owning a put option on the firm with exercise price F.
3. Due to pay F in interest and principal next period.

In terms of call options, debt was equal to

1. Owning the firm.
2. Selling a call option on the firm with exercise price F.

Rewrite the put call parity

$$C_0 = P_0 + S_0 - \frac{X}{1+r_f}$$

as

$$-C_0 + S_0 = \frac{X}{1+r_f} - P_0$$

We thus find that debt can be rewritten as:

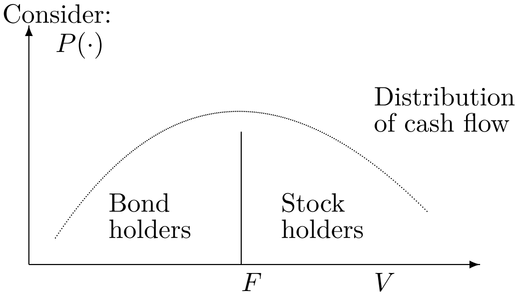
1. Having due F in interest and principal next period.
2. Selling a put option with exercise price F.

Thus, we can write the value of debt and equity in terms of both put and call options.

13.6 Probability distribution of firm value.

It us useful to remember the underlying distribution of the value of the firm, and how that affects the value of equity and debt.

Consider:



The equityholders will only get paid if the value of the firm next period is greater than F, the bond payments. Thus, they care about the probablity that V_t is greater than F, the area to the right in this picture.

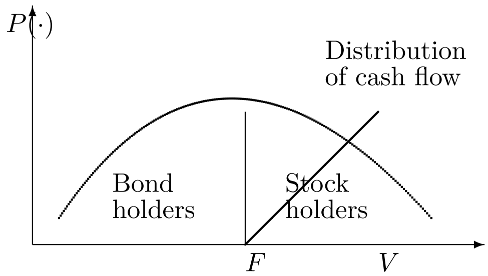
Can such thinking be used to price things.

Yes, that is the thinking behind real options theory.

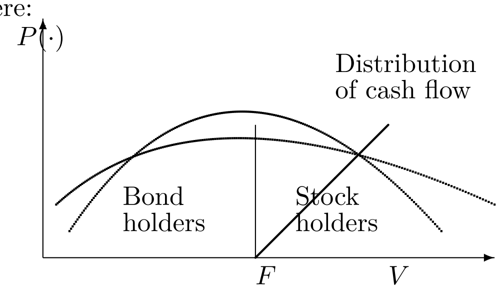
13.7 Negative NPV projects.

Let us now look at a claim we looked at earlier, that it may pay for the owner of a firm to invest in negative NPV investments, if these were *very risky*. At the time, this may have been mysterious, but let us reconsider this statement with the insights from option pricing theory.

Assume the current probability distribution of the firm value looks like:



By taking on a negative NPV, high risk project, the distribution of the firm value may change as shown here:



Here, the equity holders only care about the cases where the value of the firm is greater than the promised bond payment.

By taking on the risky project, the equityholders increase the probability that they may receive a payment at the cost of debtholders. The area to the right of the bond payment F in the figure incrases.

If you recall the definition of equity as a call option on the firm, and what we have seen earlier in the context of the Black-Scholes model, that the value of a call will increase if the volatility of the firm increases, it is easy to understand why equityholders may take on risky negative NPV projects. By taking on this project, they increase the volatility of the underlying (the firm value.) This increases the price of equity, which can be interpreted as a call price.

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