Financial Derivatives for Corporate Risk Management

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Lecture overview

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- Options
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1 Introduction

The unbelievable growth in the amount of derivatives contracts outstanding a major contribution of finance to society. It allows the hedging of risks that could before not be insured against.

In this lecture we look at the corporate use of *derivatives*.

Derivatives are defined as: Securities traded in financial markets that whose value depend on the price/value of some observable(contractible) financial asset.

To motivate, consider a very common corporate situation.

2 Example: Hedging currency risk

Exercise 1.

You are the CFO of a US-based corporation. It is now the end of February '20, and you have just signed a contract that will pay Euros (EUR) 1 mill one year from now. If you do nothing, you receive next March (in dollars)

$$\text{EUR 1mill} \times \text{Spot exchange rate}$$

The current exchange rate S_t is 1.09. The spot exchange rate one year from now S_T is uncertain, but there are ways to do something about this uncertainty.

- You can enter a forward contract. Looking at the CME, you find the following *Euro FX Futures Quotes*

Maturity	Forward Price
Jul '20	1.0968
Sep '20	1.1005
Dec '20	1.1056
Mar '21	1.1108

If you enter into a forward contract at these quotes, you commit yourself to exchanging the EUR 1 million at the forward rate in March '21.

- You can trade an option. At the CME you will also find quotes of FX options EUR/USD with maturity February 2021.

Option Price	Strike quote
37200	1.0900
34100	1.0950
31200	1.1000
28500	1.1050
26000	1.1100

The option price is the USD price for a option for 1 mill EUR with the indicated strike quote. The way the option works, you can choose to exchange the 1 mill EUR at the indicated strike. To enter into the option contract, you have to pay the premium up front.

1. Suppose you enter into a Mar '21 forward contract for 1 mill EUR, which guarantees that the USD amount you get for 1 mill EUR is 1 mill $\times$ 1.1108.
   (a) Suppose the spot exchange rate in March '21 is 1.05. What would your position have been without the forward contract? With the forward contract?
   (b) Suppose the spot exchange rate in March '21 is 1.15. What would your position have been without the forward contract? With the forward contract?
   (c) Is there a general relation between your position and the spot exchange rate in March '21?
2. Suppose you instead buy the option with a strike price of 1.1.
   (a) Suppose the spot exchange rate in March '21 is 1.05. What is the value of the option contract? What is your total position?
   (b) Suppose the spot exchange rate in March '21 is 1.15. What is the value of the option contract? What is your total position?
   (c) Is there a general relation between your position and the spot exchange rate in March '21?

Solution to Exercise 1.

Lets look at a forward contract.

One year forward, rate 1.1108:

Commit to exchanging 1 mill EUR to USD at the forward exchange rate, which means you guarantee that one year from now, you receive

$$\text{USD } 1,110,800$$

Illustrate outcomes of forward contract.

Suppose exchange rate next year falls to 1.05?

If you didn't have the commitment of the forward, would be getting

$$\text{EUR 1 mill } \times 1.05 = 1,050,000 \text{ USD}$$

instead of the committed

$$\text{USD } 1,110,800$$

In this case hedging has produced a (ex post) gain of

$$\begin{aligned} & \text{EUR 1 mill } \times (1.1108 - 1.05) \\ = & \quad 60,800 \text{ USD} \end{aligned}$$

On the other hand, suppose the exchange rate S_T increases to 1.15?

Then, if did not have the forward, would have gotten

$$\text{USD } 1,150,000$$

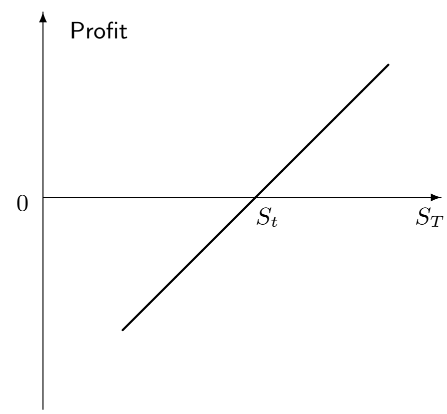
but with the forward, gets USD 1,110,800. This is a loss of $\text{USD } 1,110,800 - 1,150,000 = 39,200$ relative to not using a forward contract.

This illustrates that hedging does not necessarily guarantee against losing, but it does give the hedger a *predictable* future cashflow.

With forwards and futures, participate in both directions (losses and gains).

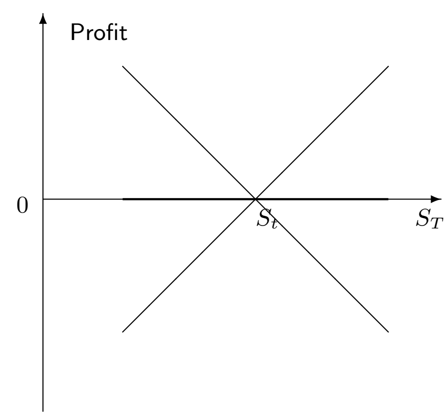
Think about this in terms of diagrams.

The promised payment of EUR 1 mill can be viewed as the "underlying." (Let S_t be the exchange rate). Your position looks like this.



If the exchange rate falls, you loose.

If you hedge with forwards/futures, you combine a short forward with your underlying.



The net effect is to make your portfolio predictable.

Let us next consider the option case. The option allows you to choose whether to use it at the exchange rate of 1.1.

Suppose exchange rate next year falls to 1.05

With that spot rate, would have gotten

$$\text{EUR 1 mill} \times 1.05 = 1,050,000 \text{ USD}$$

The option gives you the right to exhange at the rate of 1.1, will receive

$$\text{EUR 1 mill} \times 1.1 = 1,100,000 \text{ USD}$$

Having the option results in a gain of

$$1,100,000 - 1,050,000 = 50,000$$

Your total position is $1,100,000$.

On the other hand, suppose the exchange rate S_T increases to 1.15?

With that spot rate, would get

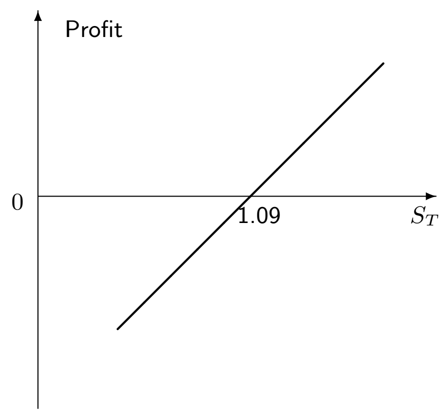
$$\text{EUR 1 mill} \times 1.15 = 1,150,000 \text{ USD}$$

The option gives you the possibility of translating at an exchange rate of 1.1.

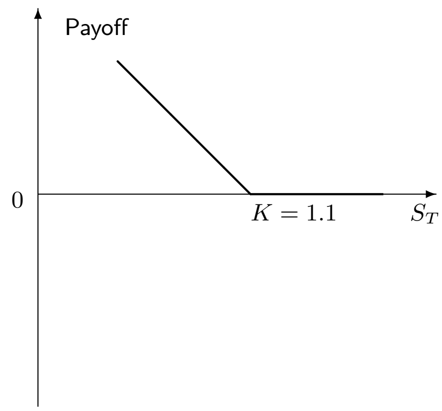
But why should you? The value of the option at the exercise date is zero.

Your total position is 1,150,000.

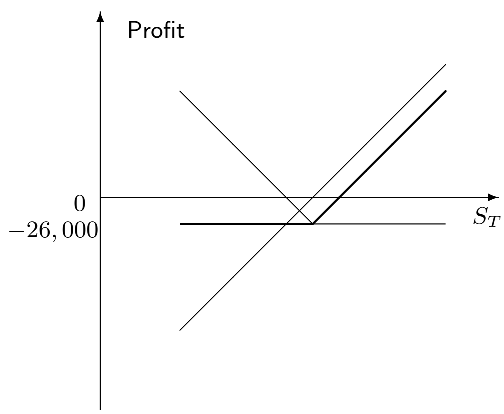
Suppose in this example that you are mainly concerned with unfavorable currency movements (the downside risk) (Have to meet payments of subcontractors etc), may want to participate in favorable ones, which means you want to protect against currency depreciation.



If you think about payoffs of options contract, to offset downside, want a contract with positive payoff when $S_T < 1.1 = K$.



This is the payoff for a put option.
To reduce downside risk, buy options, strike 1.1, and amount of EUR 1 mill.
But to get the options, need to pay a premium. Total profit profile looks like.



3 Forward contracts

Example

You go to the bookstore, look for a textbook. The textbook is out of stock. The clerk puts it on order for you, it will cost you $50 at delivery. Congratulations, you have just entered a forward contract.

Note that with a forward contract there are no choices involved. The item *must* be delivered at the given price. The only case where a delivery is not made is when one of the parties can not fulfill its obligations (bankruptcy).

Define forward contracts as agreements to buy (sell) given *amounts* of some asset (typically called the *underlying* asset) at given *price* (forward price) and at given *time* (expiry date).

To any forward contract there are two parties:
"Long": Party buying commodity in the future (buy forward)
"Short": Party selling commodity in the future (sell forward)
Contract specifies:

- Amount and other properties of good to be delivered.
- Forward price (F)
- Time of delivery (T)
- Where and how delivery is to take place.

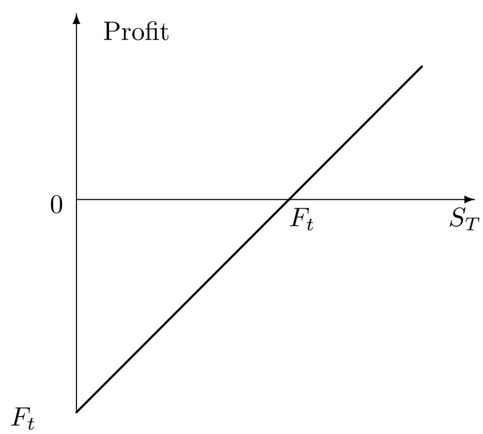
Note

- Each forward contract has both a buyer and a seller, forward contracts are in zero net supply, i.e. Forwards are pure *risk-sharing* devices.
- Usually, no money changes hands until the final date.
- The forward price is set to achieve this.

4 Forward/Futures - relation to underlying

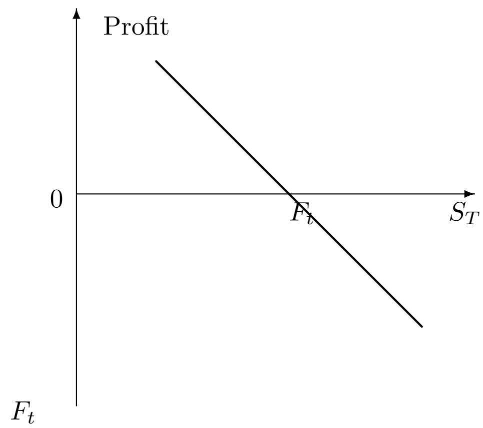
The forward is an agreement at time t to trade at time T using a forward price F_t. What is the value of your position when the contract mature?

If the forward agreements specifies that you should *buy* the underlying asset for the forward price (Long forward), the value of the contract is the difference between this price and what you could otherwise have bought the good for, the current spot price S_T.



You gain if $S_T > F_t$, lose if $S_T < F_t$.

On the other hand, if the forward agreement involves *selling* the asset (Short forward), your position is the opposite,



You Gain if $S_T < F_t$, Loose if $S_T > K$

5 Pricing of a forward contract

Pricing of a forward contract is a straightforward application of the no-arbitrage principle.

Key to pricing is to realize that one can achieve the same cashflows as those of a forward contract by other means.

For example, a forward contract to sell a stock: If you are the seller, you can buy the stock now, and deliver the stock at the forward date. To avoid any payment now, you borrow the current price of the stock, and pay back the borrowing at the forward date.

Exercise 2.

Consider a forward contract on an underlying asset that provides no income. There are also no restrictions on shortselling of the underlying asset. Let S_t and S_T denote the price of the underlying asset at t and T, respectively. r denotes the riskfree rate. Then the (time-t) forward price F_t for a contract with deliver date T has to satisfy

$$F_t = S_t(1+r)^{(T-t)},$$

i.e., the forward price is the future value of the current price of the underlying.

Use arbitrage arguments to show this.

Solution to Exercise 2.

It is easy to show that violations of this will lead to free lunches. Let us start with the case where

$$F_t > S_t(1+r)^{(T-t)}$$

Table 1 illustrates how we would set up a portfolio to exploit this free lunch.

Table 1 Arbitrage strategy for case $F_t > S_t(1+r)^{(T-t)}$

	Time	
	t	T
Sell forward	0	$F_t - S_T$
Borrow S_t	S_t	$-S_t(1+r)^{(T-t)}$
Buy underlying	$-S_t$	S_T
Total	0	$F_t - S_t(1+r)^{(T-t)} > 0$

On the other hand, if $F_t < S_t(1+r)^{(T-t)}$, it is also easy to exploit the free lunch, as table 2 illustrates

Table 2 Arbitrage strategy for case $F_t < S_t(1+r)^{(T-t)}$

	Time	
	t	T
Buy forward	0	$S_T - F_t$
Invest S	$-S_t$	$S_t(1+r)^{(T-t)}$
Short underlying	S_t	$-S_T$
Total	0	$S_t(1+r)^{(T-t)} - F_t > 0$

To avoid arbitrage we need an exact inequality

$$F_t = S_t(1+r)^{(T-t)}$$

6 Option Definitions

A *Call option* is a right to *buy* a underlying security at a fixed price (exercise price - K) in some given time period (expiry).

A *Put option* is the right to *sell* a underlying security at a fixed price (exercise price) in some given time period.

If we *use* the option to buy/sell the asset we *exercise* the option.

If the option is an *European* option, it can only be exercised at the expiry date.

If the option can be exercised any time up to the expiry date, it is called an *American* option.

7 Option payoff/profit diagrams

7.1 Payoff at maturity. Position diagrams.

We will start by looking at a useful tool for analysing options, the *position diagram*. These describes the payoff of an option (or some other security) at its maturity.

Exercise 3.

Consider buying either put or call options on IBM stock with an exercise price of $X = 100$. The current price of IBM stock S_0 is 100.

1. What are the cash flows at maturity (time T) if the then stock price (S_T) is 120?
2. What are the cash flows at maturity (time T) if the then stock price (S_T) is 80?
3. Plot the payoffs from the call and put options as functions of stock price at maturity.

Solution to Exercise 3.

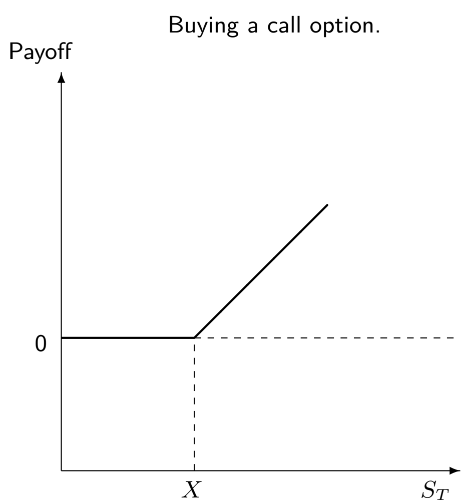
1. What if the stock price S_T at maturity is \$120?
    - If you hold a call option:
        - If you use the option, you will buy stock at the exercise price $X = 100$.
        - This stock can then be sold immediately at the current stock price $S_T = 120$.
        - Payoff if you hold a call option: $S_T - X = 120 - 100 = \$20$.
    - What if you hold a put option?
        - The option gives you the right to sell stock at $X = 100$.
        - If you don't exercise the option, you can sell stock at todays price $S_T = 120$.

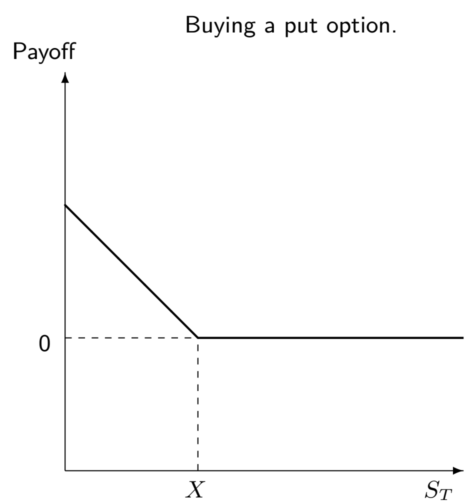
- It does not pay to exercise option.
- Payoff of holding a put option: \$0.

2. What if the stock price turned out to be lower than the exercise price, $S_T = 80$?
   - If you hold a call option:
     - The option gives you the right to buy stock at exercise price $X = 100$.
     - Alternatively, you can buy stock at current price $S_T = 80$.
     - Hence, it does not pay to exercise option.
     - Payoff \$0.
   - What if you hold a put option, and the stock price $S_T = 80$?
     - You can use the option to sell stock at $X = 100$.
     - In the market you buy stock at todays price $S_T = 80$.
     - Payoff from put option: $X - S_T = 100 - 80 = 20$.
3. To make the plot observe that we can summarize the payoffs at maturity from buying options as:
   - Call option: Payoff $= \max(0, S_T - X)$.
   - Put option: Payoff $= \max(0, X - S_T)$.

   Note that the payoff of an option is never negative.

   These formulas can be summarized as shown in the figures below. These show the payoffs at maturity when buying options.





We can summarize the payoffs at maturity from buying options as:

- Call option: Payoff $= \max(0, S_T - X)$.
- Put option: Payoff $= \max(0, X - S_T)$.

Note that the payoff of an option is never negative.

These were the pictures when you bought options. Let us next look at the opposite situation. The fact that someone is buying the option means that someone is selling it. What are the payoffs at maturity for the seller of an option?

Exercise 4.

Consider selling either put or call options on IBM stock with an exercise price of $X = 100$. The current price of IBM stock S_0 is 100.

1. What are the cash flows at maturity (time T) if the then stock price (S_T) is 120?
2. What are the cash flows at maturity (time T) if the then stock price (S_T) is 80?
3. Plot the payoffs from the call and put options as functions of stock price at maturity.

Solution to Exercise 4.

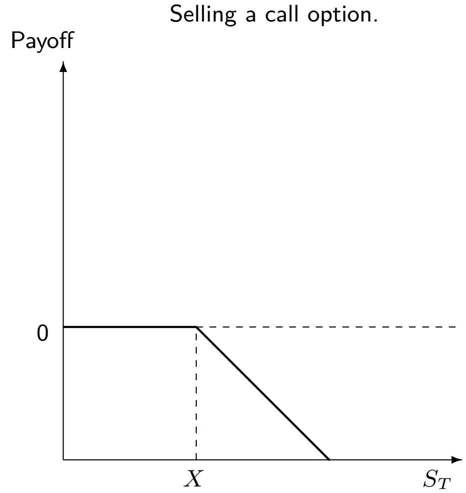
1. What if the stock price S_T at maturity is \$120?
   - If you sold a call option:
     - If the option is exercised, you need to supply the stock, getting the exercise price $X = 100$.
     - To supply the stock you need to pay $S_T = 120$ in the market.
     - Payoff from selling a call option: $X - S_T = 100 - 120 = \$ - 20$.
   - What if you sold a put option?
     - It does not pay to exercise option.
     - Payoff from selling a put option: \$0.
2. What if the stock price turned out to be lower than the exercise price, say $S_T = 80$?
   - If you sold a call option:
     - It does not pay to exercise option.
     - Payoff \$0.
   - What if you sold a put option, and the stock price $S_T = 80$?
     - You have to buy stock at the exercise price $X = 100$.
     - In the market you can sell this stock at todays price $S_T = 80$.
     - Payoff from put option: $S_T - X = 80 - 100 = -20$.

For the seller of options, the payoffs can be summarized as:

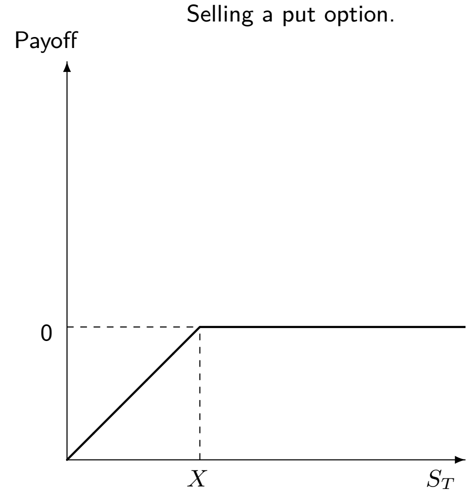
- Call option: Payoff $= \max(0, S_T - X)$.
- Put option: Payoff $= \max(0, X - S_T)$.

The figure below shows the payoffs (at maturity) from selling options.

Selling a call option.



Selling a put option.



7.2 Total profits from holding options.

In the previous we looked at what the happened at the maturity date. We saw that a call(put) option will only be exercised if the price of the underlying security is larger(smaller) than the *exercise* price.

We next look at look at the total profit from holding an option.

Remember that to buy an option, be it put or call, you have to pay a price on that option.

Thus, to get the total profit from holding an option contract, we need to subtract the premium.

- Call option: Profit $= \max(0, S_T - X) - C_0$.
- Put option: Profit $= \max(0, X - S_T) - P_0$.

Exercise 5.

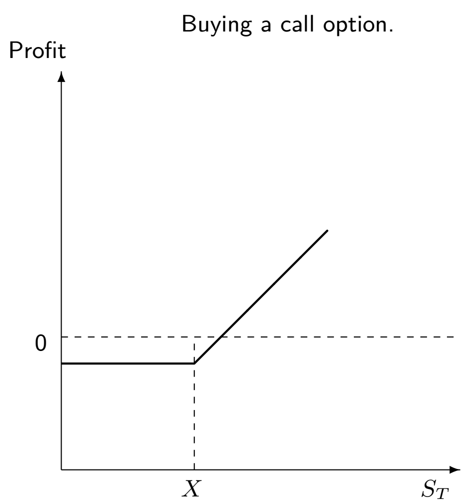
Put and call options on IBM stock with an exercise price of $X = 100$ are traded at $3 and $5, respectively. The current price of IBM stock S_0 is 100.

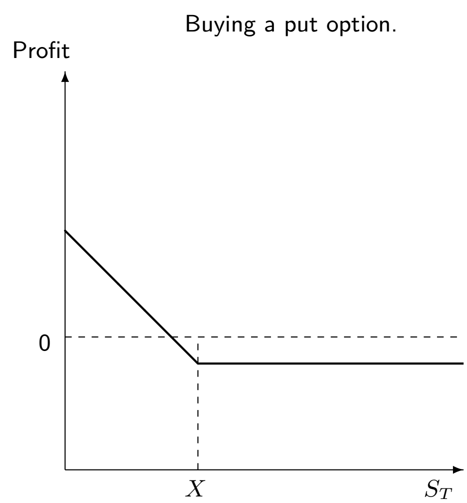
1. What is the total cashflow from the transaction of buying the call if the price of the underlying at maturity turns out to be 120.
2. What if the price of the underlying is 80?
3. Plot the total cash flows from buying call and put options.

Solution to Exercise 5.

1. If you buy the call you pay a premium of $5 on the call option, and the stock price turns out to be $120 at maturity. Then your payoff at maturity was $20, but your *total profit* from that contract is $20 - 5 = 15$.
2. On the other hand, if the stock price turned out to be $80, your call option would expire worthless, and your total profit would be $0 - 5 = -5$.
3. Generally, to get the total profit from holding an option contract, we need to subtract the premium:
   - Call option: Profit $= \max(0, S_T - X) - C_0$.
   - Put option: Profit $= \max(0, X - S_T) - P_0$.

   The figure belows diagrams the dollar profit from buying a number of financial contracts.





Let us also consider what happens if we *sell* a security.

If you sell an option, you get the premium up front, but as we saw above, if the option is exercised, you will have a negative payoff at maturity.

Thus, for the seller of options, the total profits can be summarized as:

- Call option: $\text{Payoff} = C_0 + \max(0, S_T - X)$.
- Put option: $\text{Payoff} = P_0 + \max(0, X - S_T)$.

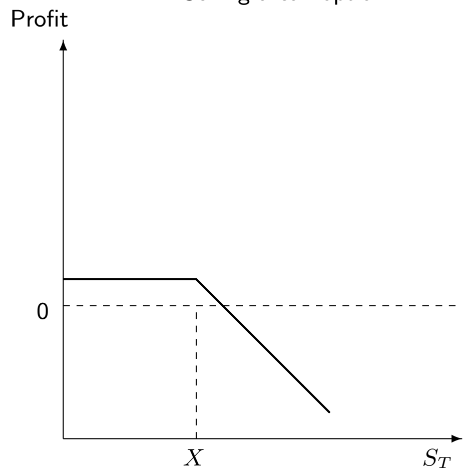
Exercise 6.

Put and call options on IBM stock with an exercise price of $X = 100$ are traded at $3 and $5, respectively. The current price of IBM stock S_0 is 100.

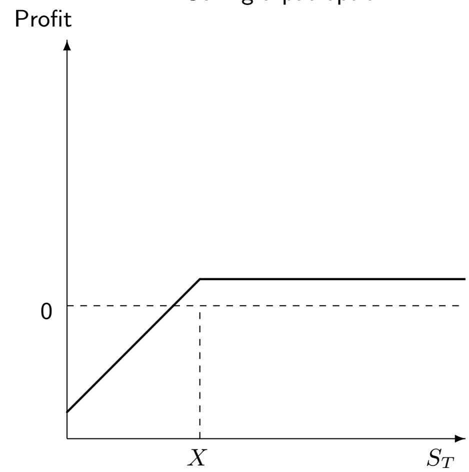
1. Plot the total cash flows from selling (issuing) these call and put options.

Solution to Exercise 6.

Selling a call option.



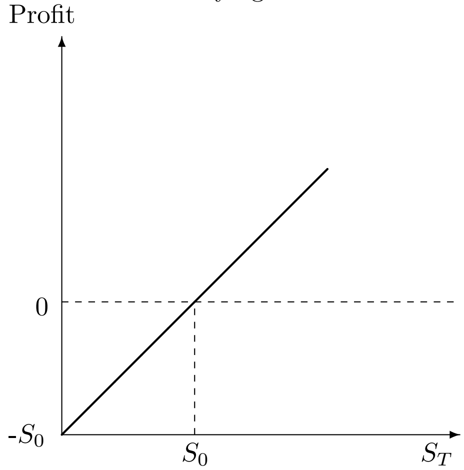
Selling a put option.



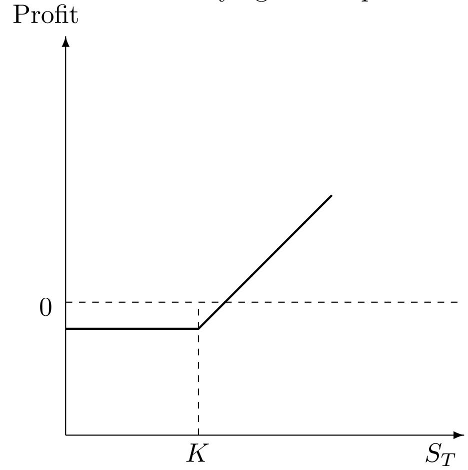
7.3 Comparing options with other securities

Profit from buying financial contracts.

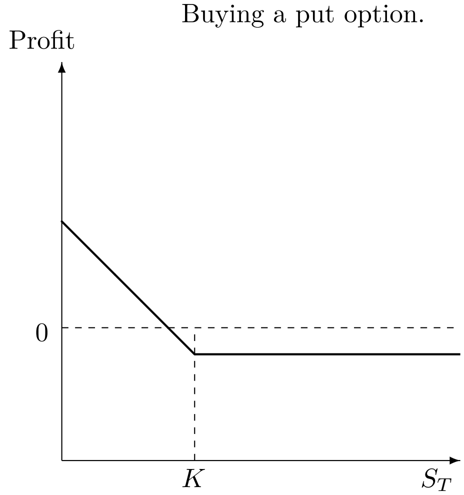
Buying a stock



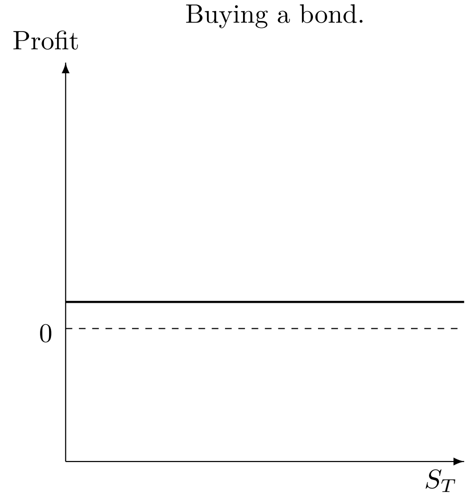
Buying a call option.



Buying a put option.

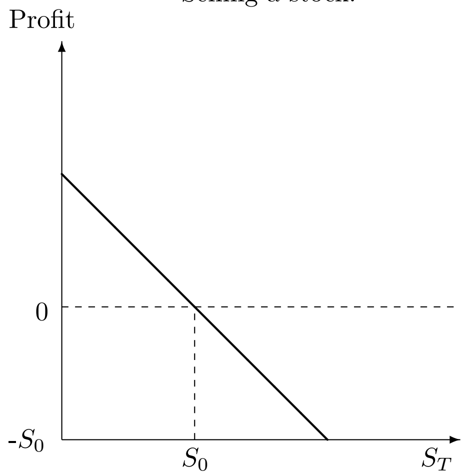


Buying a bond.

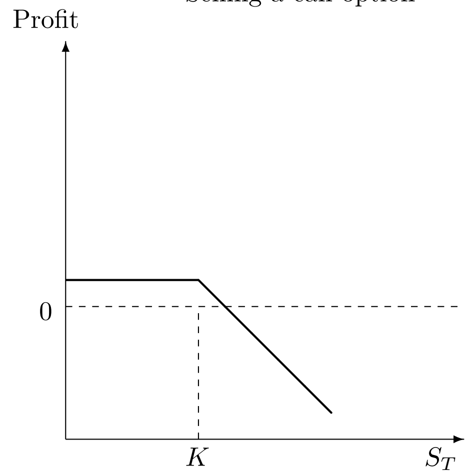


Profit from selling financial contracts.

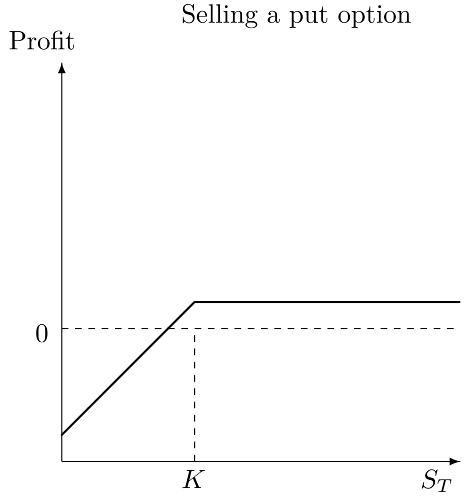
Selling a stock.



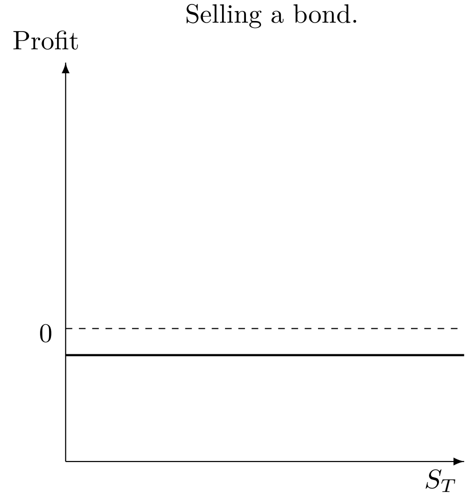
Selling a call option



Selling a put option



Selling a bond.



8 On pricing of options

The ability to price contingent claims such as options and other derivative securities is a recent innovation in Finance. Indeed, the existence of a simple formula for the price of an option (The Black-Scholes formula) was one of the chief reasons for the explosive growth in these markets. The CBOE (Chicago Board of Options Exchange) started trading options on common stock in 1973, and in the same year two important papers describing option pricing formulas were published: Black and Scholes (1973) and Merton (1973)

The type of analysis used to find option values has had impact in most areas of finance. All relies on a *no-arbitrage* argument, which we can summarise as

If two portfolios or assets have the same payoffs tomorrow, they must have the same price today.

The challenge in coming up with an option pricing formula is the *contingent* feature of options, they are only used (excercised) when it results in a positive payoff.

This is different from e.g. pricing a forward. To price a forward contract, one can set up a portfolio which will replicate the payoffs from the forward using a position in the underlying security and riskfree borrowing/lending. This replicating portfolio only needs to be set up once, and it then works perfectly at the expiry of the forward.

An options contract can also be replicated using the underlying security and riskfree borrowin/lending, but the replicating portfolio needs to be changed as time passes (and the price of the underlying security changes).

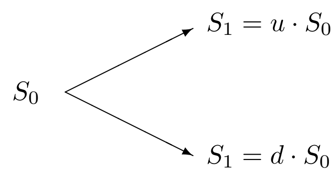
Constructing the replicating strategy is a nontrivial mathematical problem.

To help in understanding how option pricing works, we do this construction in a simple setting, the *binomial* framework.

In practical use, we use a more complex algorithm, the Black Scholes formula.

8.1 Stock Price movement.

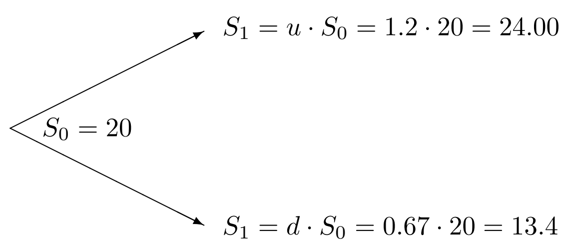
The simplest assumption, which we will use from now on, is that the stock price next period can take on only two values,



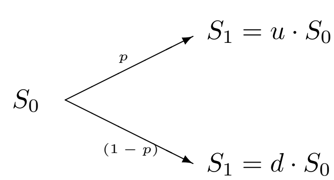
it can increase to $u \cdot S_0$ or decrease to $d \cdot S_0$.

Example

$$\begin{aligned} S_0 &= 20 \\ u &= 1.2 \\ d &= 0.67 \\ r_f &= 10\% \end{aligned}$$



Also assume that the price jumps up with probability p and jumps down with probability $(1 - p)$.



Then

$$
\begin{aligned}
E[\tilde{S}_1] &= p \cdot uS_0 + (1-p) \cdot dS_0 \\
&= (pu + (1-p)d)S_0
\end{aligned}
$$

Example

Continuing our example, with $S_0 = 20$, $u = 1.2$ and $d = 0.67$.

$$p = \frac{1}{2} \quad \Rightarrow \quad E[S_1] = 18.70$$

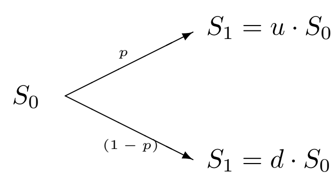
$$p = 0.625 \quad \Rightarrow \quad E[S_1] = 20$$

$$p = \frac{2}{3} \quad \Rightarrow \quad E[S_1] = 20.40$$

The major surprise about pricing options is that the expected stock price is not relevant for the price of a call option on the stock.

8.2 Call option payoffs.

We now have the stock price movement

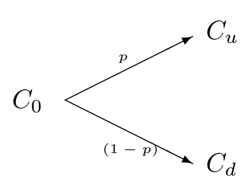


What will be the payoff at time 1 of a call option with exercise price X maturing at time 1?

$$C_1 = \max(0, S_1 - X)$$

Example

In our example, if $X = 20$, the payoffs at maturity is



where

$$
\begin{aligned}
C_u &= \max(0, S_1 - X) \\
&= \max(0, u \cdot S_0 - X) \\
&= \max(0, 24 - 20) = 4 \\
C_d &= \max(0, S_1 - X) \\
&= \max(0, d \cdot S_0 - X) \\
&= \max(0, 13.40 - 20) = 0
\end{aligned}
$$

8.3 Constructing a hedge portfolio.

In order to find the call price at time 0, we now ask the following question: What combination of stock and call options will give us a riskless payoff next period? More specifically: If we buy one stock, how many call options do we need to buy/sell to make the payoff next period riskless?

Let m be the number of call options we buy. The payoff from a strategy of buying 1 stock and m call options should be the same in both the up-state and the down-state, or

$$mC_u + S_u = mC_d + S_d$$

$$\Rightarrow \quad mC_u + uS_0 = mC_d + dS_0$$

$$\Rightarrow \quad m(C_u - C_d) = (d-u)S_0$$

$$\Rightarrow \quad m = \frac{(d-u)S_0}{(C_u - C_d)}$$

In our example

$$\begin{aligned} m &= \frac{(0.67-1.2)20}{4-0} \\ &= -0.1325 \cdot 20 \\ &= -2.65 \end{aligned}$$

Let us check that buying 1 stock and selling 2.65 call options give the same payoffs in both states:

State:		u	d
Buy stock:		24	13.40
Sell 2.65 Call:		$-2.65 \cdot 4 = -10.60$	0
Total payoff:		13.40	13.40

Thus, the strategy of buying 1 stock and buying -2.65 calls will next period have a known payoff, and hence a certain (known) return. This return has to be equal to risk free rate of return, r_f, to avoid arbitrage.

The cost of the portfolio is $S_0 + mC_0$.

In our example, the cost is $20 - 2.65C_0$, giving a return on the portfolio of

$$\frac{13.40}{20 - 2.65C_0} - 1$$

This return has to be equal to the riskfree rate $r_f = 10\%$, and we can solve for C_0.

$$\frac{13.40}{20 - 2.65C_0} = 1 + r_f$$

$$20 - 2.65C_0 = \frac{13.40}{1.1}$$

$$C_0 = \frac{20 - \frac{13.40}{1.1}}{2.65} = 2.95$$

The price of the call is $C_0 = 2.95$.

Let us check that the return on the portfolio is equal to the riskfree rate:

$$\begin{aligned} \text{Return} &= \frac{13.40}{20 - 2.65 \cdot 2.95} - 1 \\ &= 1.0999 - 1 \approx 10\% \end{aligned}$$

8.4 Developing a binomial pricing "cookbook" (Risk-neutral probabilities)

Now go back to the derivation of the call price. The investment of $(S_0 + mC_0)$ had a riskless payoff next period, which had to be equal to the payoff you would get if you invested the same amount at the riskfree rate:

$$uS_0 + mC_u = (S_0 + mC_0)(1 + r_f)$$

$$\Rightarrow \quad S_0 + mC_0 = \frac{uS_0 + mC_u}{1 + r_f}$$

$$\Rightarrow \quad C_0 = \frac{uS_0 + mC_u - S_0(1 + r_f)}{m(1 + r_f)}$$

$$\Rightarrow \quad C_0 = \frac{mC_u + S_0[u - (1 + r_f)]}{m(1 + r_f)}$$

Substitute

$$m = \frac{(d-u)S_0}{C_u - C_d}$$

in the above equation and simplify

$$\begin{aligned}
C_0 &= \frac{(\frac{(d-u)S_0}{C_u - C_d})S_0 C_u + S_0[u - (1 + r_f)]}{(\frac{(d-u)S_0}{C_u - C_d})(1 + r_f)S_0} \\
&= \frac{C_u(d-u) + (C_u - C_d)[u - (1 + r_f)]}{(d-u)(1 + r_f)} \\
&= \frac{C_u[(d-u) + u - (1 + r_f)]}{(d-u)(1 + r_f)} \\
&\quad - \frac{C_d[u - (1 + r_f)]}{(d-u)(1 + r_f)} \\
&= \frac{C_u[(d - (1 + r_f)] - C_d[u - (1 + r_f)]}{(d-u)(1 + r_f)} \\
&= \frac{C_u(\frac{d-(1+r_f)}{d-u}) - C_d(\frac{u-(1+r_f)}{d-u})}{1 + r_f} \\
&= \frac{C_u(\frac{(1+r_f)-d}{u-d}) + C_d(\frac{u-(1+r_f)}{u-d})}{1 + r_f}
\end{aligned}$$

Define

$$\begin{aligned}
q &= \frac{(1 + r_f) - d}{u - d} \\
1 - q &= \frac{u - (1 + r_f)}{u - d}
\end{aligned}$$

Then

$$C_0 = \frac{qC_u + (1 - q)C_d}{1 + r_f}$$

The value q we calculated above is called the 'risk-neutral probability' or alternatively the 'hedging probability.'

This gives us the following "cook book" for pricing options in a binomial world:

1. Calculate the 'risk-neutral' probabilities
$$q = \frac{(1 + r_f) - d}{u - d}$$
and
$$1 - q = \frac{u - (1 + r_f)}{u - d}$$
2. Find the payoffs in each state:
$$C_u = \max(0, uS_0 - X)$$
$$C_d = \max(0, dS_0 - X)$$
3. Calculate the call price.
$$C_0 = \frac{qC_u + (1 - q)C_d}{1 + r_f}$$

Example

$$\begin{aligned} S_0 &= 20 \\ u &= 1.2 \\ d &= 0.67 \\ r_f &= 10\% \\ X &= 20 \end{aligned}$$

Then

$$\begin{aligned} q &= \frac{(1 + r_f) - d}{u - d} \\ &= \frac{(1 + 0.10) - 0.67}{1.2 - 0.67} = 0.81 \\ 1 - q &= \frac{u - (1 + r_f)}{u - d} \\ &= \frac{1.2 - (1 + 0.10)}{1.2 - 0.67} = 0.19 \end{aligned}$$

and

$$\begin{aligned} C_u &= \max(0, uS_0 - X) \\ &= \max(0, 1.2 \cdot 20 - 20) \\ &= \max(0, 4) = 4 \\ C_d &= \max(0, dS_0 - X) \\ &= \max(0, 0.67 \cdot 20 - 20) \\ &= \max(0, -6.6) = 0 \\ C_0 &= \frac{qC_u + (1 - q)C_d}{1 + r_f} \\ &= \frac{0.81 \cdot 4 + 0.19 \cdot 0}{1.1} = 2.95 \end{aligned}$$

8.5 Put options

Knowing the price of a call option will also give us the price of a put through the put-call parity relationship.

$$P_0 = C_0 + \frac{X}{1+r_f} - S_0$$

Example

$$\begin{aligned} S_0 &= 20 \\ r_f &= 10\% \\ X &= 20 \end{aligned}$$

and we calculated

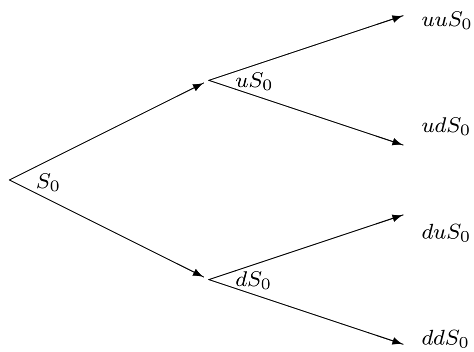
$$C_0 = 2.95$$

Then the put price is

$$\begin{aligned} P_0 &= C_0 + \frac{X}{1+r_f} - S_0 \\ &= 2.95 + \frac{20}{1.1} - 20 = 1.13 \end{aligned}$$

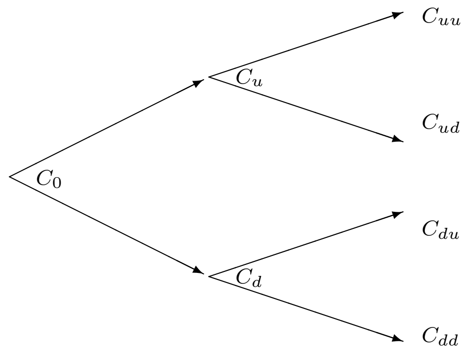
8.6 Extending binomial pricing to more than one period.

Let now the call option expire after 2 periods, $T = 2$. In each period the stock can make the same multiplicative jumps as in the one-period case:



The possible stock prices after 2 periods are uuS_0, udS_0, duS_0 and ddS_0.

To solve for option prices we go "backwards" in the tree. First we find the option prices at time 1, C_u and C_d, and then we use these call prices to find the price at time 0, C_0.



The risk-neutral probabilities stay the same,

$$
\begin{aligned}
q &= \frac{(1+r_f)-d}{u-d} \\
1-q &= \frac{u-(1+r_f)}{u-d}
\end{aligned}
$$

We use these probabilities to "roll backwards" in the tree:

$$
\begin{aligned}
C_{uu} &= \max(0, uuS_0 - X) \\
C_{ud} &= \max(0, udS_0 - X) \\
C_u &= \frac{qC_{uu} + (1-q)C_{ud}}{1+r_f} \\
C_d &= \frac{qC_{du} + (1-q)C_{dd}}{1+r_f}
\end{aligned}
$$

We then use these calculated call prices in period 1 to find the call price in period 0:

$$C_0 = \frac{qC_u + (1-q)C_d}{1+r_f}$$

Example

$$
\begin{aligned}
S_0 &= 20 \\
u &= 1.2 \\
d &= 0.67 \\
r_f &= 10\% \\
X &= 20 \\
T &= 2 \\
q &= \frac{(1+r_f)-d}{u-d} \\
&= \frac{1.1-0.67}{1.2-0.67} = 0.81 \\
1-q &= \frac{u-(1+r_f)}{u-d} \\
&= \frac{1.2-1.1}{1.2-0.67} = 0.19
\end{aligned}
$$

$$
\begin{aligned}
C_{uu} &= \max(0, S_T - X) \\
&= \max(0, uuS_0 - X) \\
&= \max(0, 28.8 - 20) \\
&= \max(0, 8.8) = 8.8 \\
C_{ud} &= \max(0, S_T - X) \\
&= \max(0, udS_0 - X) \\
&= \max(0, -3.92) = 0 \\
C_u &= \frac{qC_{uu} + (1-q)C_{ud}}{1+r_f} \\
&= \frac{0.81 \cdot 8.8 + 0.19 \cdot 0}{1.1} = 6.48
\end{aligned}
$$

Similarly

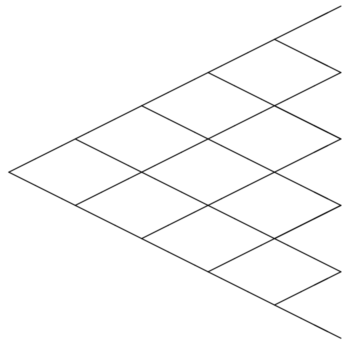
$$
\begin{aligned}
C_{du} &= \max(0, S_T - X) \\
&= \max(0, duS_0 - X) \\
&= \max(0, 16.08 - 20) \\
&= \max(0, -3.92) = 0 \\
C_{dd} &= \max(0, S_T - X) \\
&= \max(0, ddS_0 - X) \\
&= \max(0, 8.978 - 20) \\
&= \max(0, -11.02) = 0 \\
C_d &= \frac{qC_{du} + (1-q)C_{dd}}{1+r_f} \\
&= \frac{0.81 \cdot 0 + 0.19 \cdot 0}{1.1} = 0 \\
C_0 &= \frac{qC_u + (1-q)C_d}{1+r_f} \\
&= \frac{0.81 \cdot 6.48 + 0.19 \cdot 0}{1.1} = 4.77
\end{aligned}
$$

9 From the binomial towards a more realistic price assumption

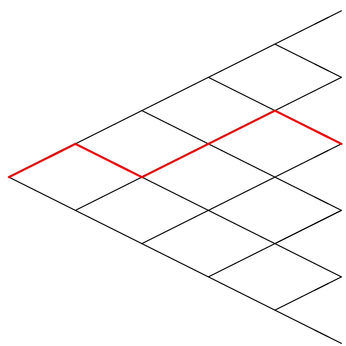
The versions of the binomial model we have seen so far will seem to have limited usefulness, having only a few possible outcomes for the underlying price process is clearly unrealistic, but for approximating the “real” solution we can increase the number of possible future price outcomes.

This is the next topic, a process for describing more realistic price “paths”

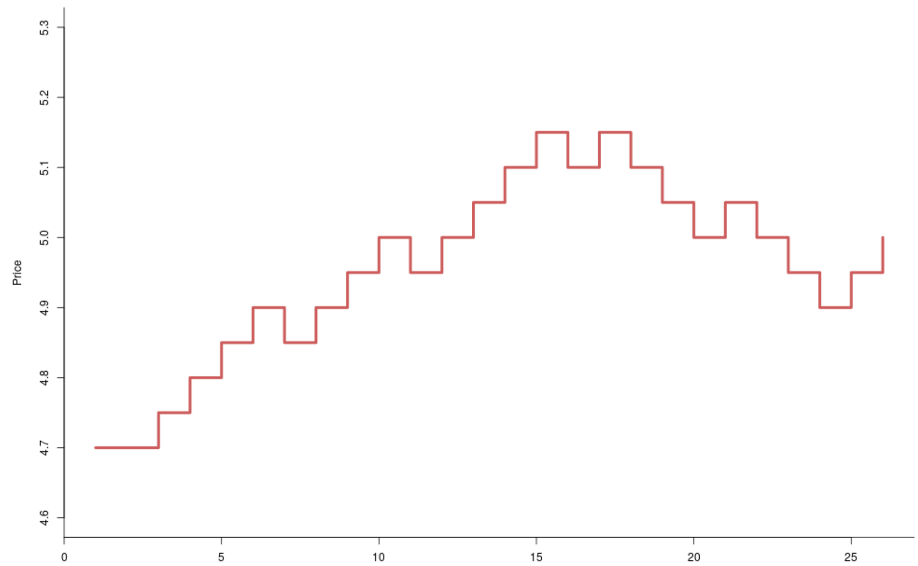
If ou increase the number of steps in a binomial tree, you increase the number of possible outcomes.



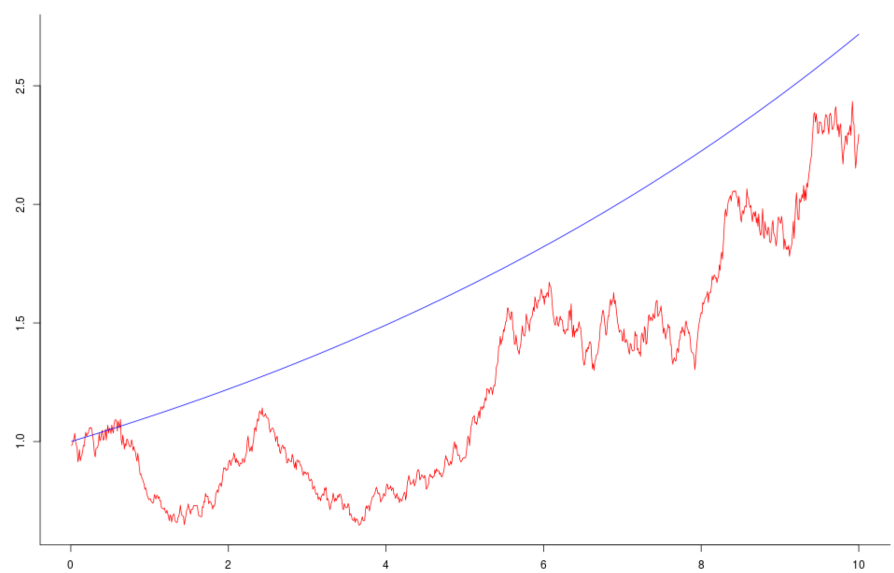
But when you look at actual “outcomes,” what the price has been, it is only one path in this tree



If we plot the history of actual prices over time, we get a picture that looks like



If you increase the number of nodes indefinitely, end up with something that looks like.



The limit, when $\Delta t \to 0$, is **Brownian Motion**. This is the object of discussion, a more realistic way of describing stock price processes.

10 The Black Scholes formula

The Black Scholes formula

The Black Scholes formula for a call option is

$$c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(\cdot) = \text{ The cumulative normal distribution}$$

The price of a put option is

$$p = Ke^{-r(T-t)} N(-d_2) - SN(-d_1)$$

11 Calculating option prices using the Black Scholes formula

Exercise 7.

Consider 3 month options with exercise prices of $K = 45$. The variance of the underlying security is $\sigma^2 = 0.20$. The risk free interest rate is $r = 6\%$. The current price of the underlying security is $S = 30$.

1. Determine the Black Scholes prices for call and put options.
2. Check that your calculations satisfy put call parity.

Solution to Exercise 7.

The Black Scholes formula for a call option is

$$c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(\cdot) = \text{ The cumulative normal distribution}$$

The price of a put option is

$$p = Ke^{-r(T-t)} N(-d_2) - SN(-d_1)$$

All except the volatility is given, the volatility is the square root of the variance

$$\sigma = \sqrt{0.20} = 0.447$$

Call

$$C_{BS}(S = 30, K = 45, r = 0.06, \sigma = 0.447214, (T-t) = 0.25)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln(\frac{30}{45}) + (0.06 + \frac{1}{2}0.20)\frac{3}{12}}{0.447\sqrt{\frac{3}{12}}}$$

$$d_1 = -1.63441$$

$$N(d_1) = 0.051$$

$$d_2 = -1.85802$$

$$N(d_2) = 0.032$$

$$C_{BS} = 0.133$$

Put prices

$$N(-z) = 1 - N(z)$$

$$p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) = 45e^{-0.06\frac{3}{12}}(1-0.032) - 30(1-0.051) = 14.46$$

Check this using put–call parity

$$c - p = S - Ke^{-r(T-t)}$$

$$p = c - S + Ke^{-r(T-t)} = 0.133 - 30 + 45e^{-0.06\frac{3}{12}} = 14.46$$

12 Using the Black Scholes pricing formula.

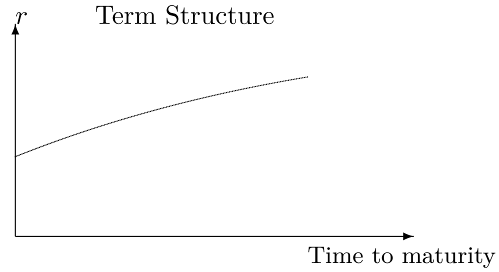
Where does the inputs to the Black Scholes model come from?

S: Look in the newspaper.

K and $T - t$ are given in the option contract.

Interest rate: Take from treasury data.

Problem: term structure of interest rate, there is no one interest rate.



Solution: Take matching maturity interest rate.

12.1 Volatility

Standard deviation (σ).

Two methods for finding estimates of volatility.

12.1.1 Historical volatility.

Given a sequence of e.g. n daily observations of the underlying,

$$\{S_n, S_{n-1}, S_{n-2}, \cdots S_1\}$$

Calculate sample standard deviation the standard way

$$\bar{S} = \frac{1}{n-1}\sum_{t=2}^{n}(\ln S_t - \ln S_{t-1})$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{t=2}^{n}(\ln S_t - \ln S_{t-1} - \bar{S})^2$$

Adjust to get annualized volatility

$$\sigma^2 = 260 \cdot \hat{\sigma}^2$$

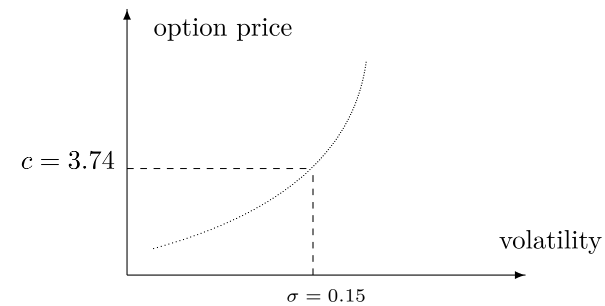
Alternatively, can use observations of the high (H_i) and low (L_i) values during a period such as a day to estimate the volatility

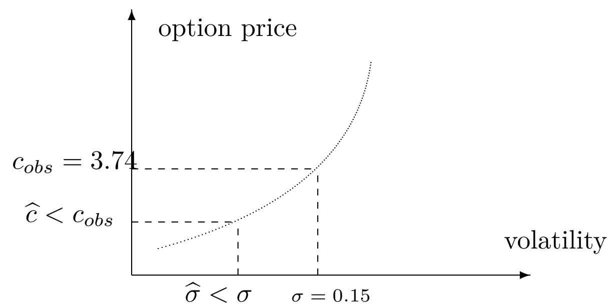
$$\hat{\sigma} = \sqrt{\frac{0.361}{n}\sum_{i=1}^{n}(\ln(H_i) - \ln(L_i))^2}$$

Both these are measures of the historical volatility of the underlying.

Instead, can find

12.1.2 Implied volatility.





Exercise 8.

Consider an option contract where the current price $S = 100$, the exercise price is $K = 100$, and time to maturity $T - t$ is one year. The risk free interest rate is 5%.

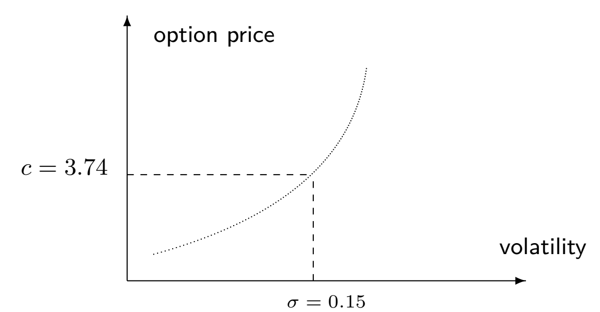
1. Suppose you observe yesterdays call price to be $C = 14.97$. What is the volatility implied in this price?

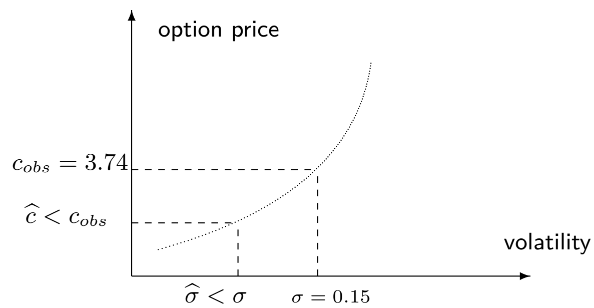
Solution to Exercise 8.

Given $C = 14.97$, can find the implied volatility by solving the equation

$$C_{obs} = 14.97.74 = C(S = 100, K = 100, \sigma, (T - t) = 1, r = 0.05)$$

Unfortunately, there is no way to analytically solve for σ in this equation, but it is easy to solve numerically.





Look in the programming notes to see a couple of ways to do this. If you do, find

$$\sigma_{implied} = 0.25$$

13 Analyzing capital structure using option theory

We will in this section look at how we can get new insights about the choice between debt and equity by seeing how they can be interpreted as options.

13.1 Defining Debt and Equity.

Consider a firm with two classes of liabilities: Equity and Debt. Assume there is a single, homogeneous class of debt with the following terms:

1. The debt is a "pure" discount bond where the firm promises to pay M dollars for each bond at the maturity date T. If there are n bonds outstanding, then the total promised payment to the debtholders is $F = nM$ at the maturity date T.
2. In the event that the firm does not make the promised payment ("default") then the firm goes bankrupt, its assets are turned over to the bondholders, and each bondholder will receive his pro rata share of the "reorganized" firm. The original equity holders will receive nothing in that event.

Let V_t denote the market value of the firm at time t. (which, by definition, will always equal the sum of the market value of debt plus equity.)

13.2 Payoff at maturity.

On the maturity date of debt:

- If the value of the firm exceeds the amount of the promised payment, (ie $V_T > F$), then it is in the interest of the equityholders (who elect management) to have debt paid. Thus, the value of the debt issue in that event will be F, and the value of equity will be $V_T - F$.
- If the value of the firm is less than the amount of the promised payment (ie $V_T < F$), then the firm can not make the promised payment. Because corporate equity enjoys limited liability, the equity holders cannot be compelled to contribute the "short fall" to pay the bondholders, and it is, clearly, not in their interest to do so. Thus, the firm will default, and the value of the debt issue in that event will be V_T, and the value of equity will be 0.

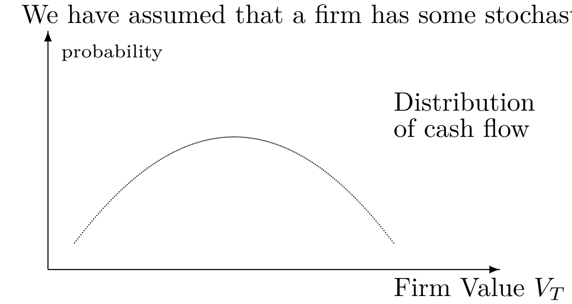
To summarize, on the maturity date:

$$\begin{aligned}
\text{Value of debt} &= \min(F, V_T) \\
&= F - \max(0, F - V_T) \\
\text{Value of equity} &= V_T - \min(F, V_T) \\
&= V_T - F + \max(0, F - V_T) \\
&= \max(0, V_T - F)
\end{aligned}$$

13.3 Interpreting the payoffs as options

Let us now try to interpret these payoffs in terms of options.

We have assumed that a firm has some stochastic value of V_T next period:



We can summarize the financing scheme as:

Bondholders

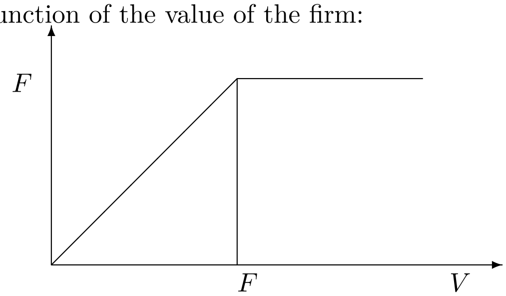
- Promised payment of F next period.
- If default occurs, bondholders own the firm.

Stockholders:

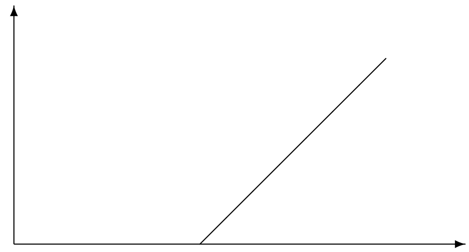
- Receive all residual cash flows after payments to bondholders.

Payments to	$V \leq F$	$V > F$
Bondholders	V	F
Stockholders	0	$V - F$
Total	V	V

We have calculated bondholders payments as $\min(V, F)$ Let us plot the payment to bondholders as a function of the value of the firm:



Similarly, stockholders payments is calculated as $\max(0, V - F)$



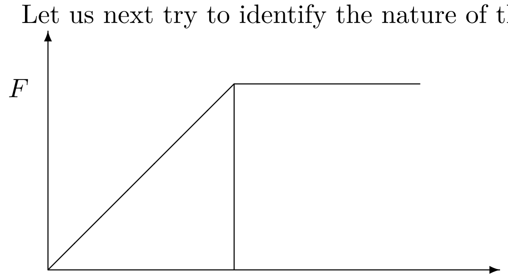
Note that payments to the stockholders and bondholders add up to the total cash flows of the firm.

$$\begin{aligned}
&\text{Value of firm}\\
&= \min(V,F) \quad \text{(Bondholders)}\\
&\quad + \max(0, V-F) \quad \text{(Stockholders)}\\
&= \begin{cases} V+0=V & \text{if } V<F \\ F+(V-F)=V & \text{if } V \geq F \end{cases}\\
&= V
\end{aligned}$$

13.4 The firm in terms of call options

Let us now see if we can identify the nature of these options. Let us start by the equity. The payoff to equity is $\max(0, V-F)$. This can be interpreted as a call option on the firm value, with exercise price F. Hence, if the firm value is less than the exercise price, do not exercise, the value of equity is zero.

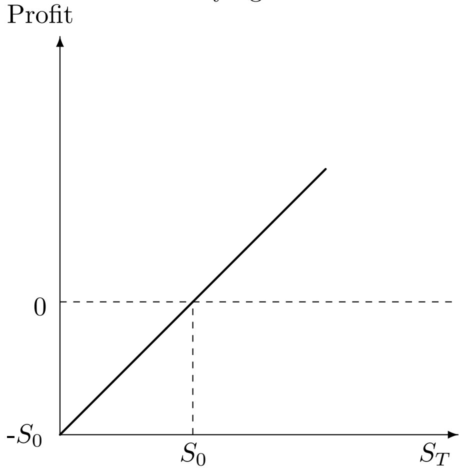
Let us next try to identify the nature of the debt.



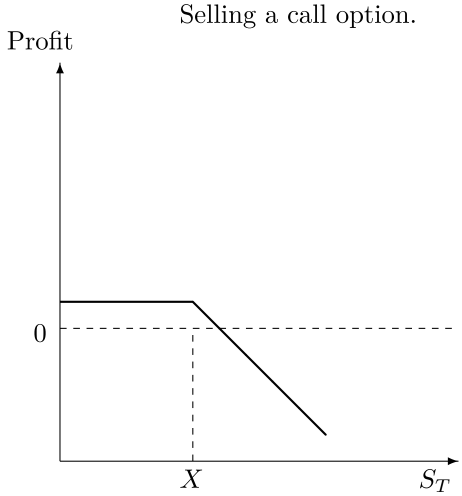
In terms of our usual stock options, we can achieve this payoff by

1. Buying the stock.
2. Selling a call.

Buying a stock.



Selling a call option.



Thus, we can view holding bonds in a firms as:

1. Owning the firm.
2. Selling a call option on the firm with exercise price F.

13.5 The firm in terms of put options

Recall the basic "put–call parity" relation which we showed earlier

$$C_0 = P_0 + S_0 + \frac{X}{1 + r_f}$$

With our interpretation of debt and equity in terms of calls, we can rewrite it in terms of puts.

We showed that equity is equal to a call with exercise price F. Using

$$C_0 = P_0 + S_0 - \frac{X}{1 + r_f}$$

By reinterpreting these in terms of the firm, we see that it is also equal to:

1. Owning the firm.

2. Owning a put option on the firm with exercise price F.

3. Due to pay F in interest and principal next period.

In terms of call options, debt was equal to

1. Owning the firm.

2. Selling a call option on the firm with exercise price F.

Rewrite the put call parity

$$C_0 = P_0 + S_0 - \frac{X}{1+r_f}$$

as

$$-C_0 + S_0 = \frac{X}{1+r_f} - P_0$$

We thus find that debt can be rewritten as:

1. Having due F in interest and principal next period.

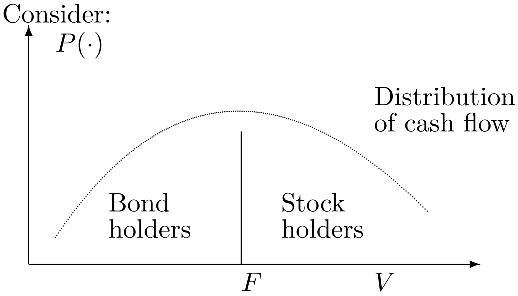
2. Selling a put option with exercise price F.

Thus, we can write the value of debt and equity in terms of both put and call options.

13.6 Probability distribution of firm value.

It us useful to remember the underlying distribution of the value of the firm, and how that affects the value of equity and debt.

Consider:



The equityholders will only get paid if the value of the firm next period is greater than F, the bond payments. Thus, they care about the probablity that V_t is greater than F, the area to the right in this picture.

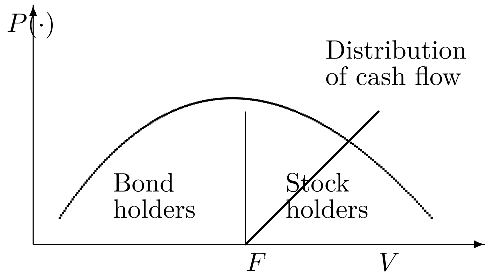
Can such thinking be used to price things.

Yes, that is the thinking behind real options theory.

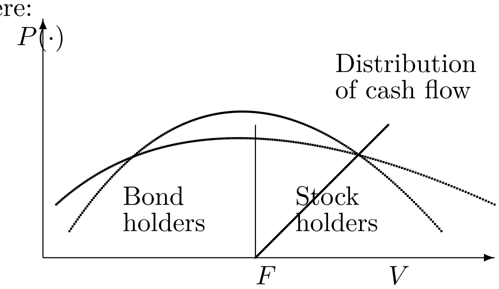
13.7 Negative NPV projects.

Let us now look at a claim we looked at earlier, that it may pay for the owner of a firm to invest in negative NPV investments, if these were *very risky*. At the time, this may have been mysterious, but let us reconsider this statement with the insights from option pricing theory.

Assume the current probability distribution of the firm value looks like:



By taking on a negative NPV, high risk project, the distribution of the firm value may change as shown here:



Here, the equity holders only care about the cases where the value of the firm is greater than the promised bond payment.

By taking on the risky project, the equityholders increase the probability that they may receive a payment at the cost of debtholders. The area to the right of the bond payment F in the figure incrases.

If you recall the definition of equity as a call option on the firm, and what we have seen earlier in the context of the Black-Scholes model, that the value of a call will increase if the volatility of the firm increases, it is easy to understand why equityholders may take on risky negative NPV projects. By taking on this project, they increase the volatility of the underlying (the firm value.) This increases the price of equity, which can be interpreted as a call price.

14 Summary – derivatives

Lecture; Tools for *hedging* of company risks.

Sources of risk: Exposure

Main tools:

- Forward Contracts
  - Fixing future value.
- Options
  - Putting a *floor* on future value
- Pricing
  - Tool: Arbitrage portfolio
  - Challenge: Constructing risk free “match” of futures/options
- Capital structure insights

References

Fisher Black and Myron S Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 7:637–54, 1973.

Robert C Merton. The theory of rational option pricing. *Bell Journal*, 4:141–183, 1973.