Lecture notes – The Capital Asset Pricing Model (CAPM)

Bernt Arne Ødegaard

30 August 2023

Contents

1 Introduction

Lecture overview:

- Intuition: What is an asset pricing model? What do we use it for?
- Equilibrium consequence of mean-variance analysis: CAPM

- Using the CAPM: Cost of capital.
- Corporate use of CAPM: Cost of equity and cost of debt

2 The Capital Asset Pricing Model (CAPM).

2.1 Introduction

Thus far our discussion has been primarily concerned with the measurement of risk and return on portfolios, given the expected returns, variances and covariances of the individual stocks in the portfolio. We now want to shift gears somewhat and talk about the way in which stocks are priced by the market. In particular, we will want to find answers to the following questions:

1. How do we measure the risk of an individual stock?
2. How is the expected of required rate of return on a stock related to its risk?

An important theoretical model that provides an answer to both of these questions is the *Capital Asset Pricing Model* (CAPM). The CAPM is widely used in practice by both the investment community for security analysis and performance measurement, and by corporate financial managers for cost of capital estimation.

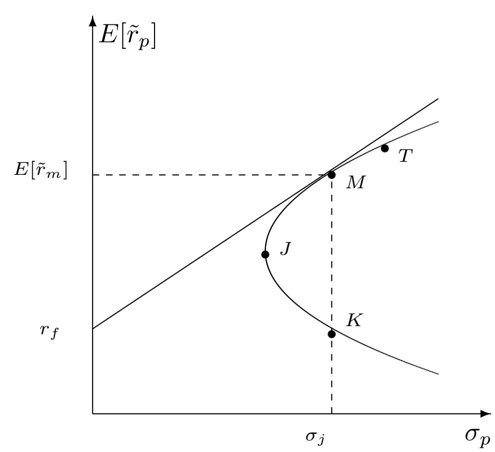
3 Assumptions of the CAPM

1. Investors choose their portfolios based on expected return and variance.
2. All investors have rational expectations and homogeneous beliefs.
3. Investors trade in perfect capital markets. This means that there are no frictions such as taxes, transaction costs, or restrictions on short sales.

These assumptions are clearly unrealistic, as are most assumptions that underlie nearly all theories in the social and physical sciences. However, it would be unfair to judge a theory based solely upon the assumptions it makes. The appropriate test of the usefulness of a theory is how well its predictions match the actual empirical data. As we will see, the CAPM does a reasonable job of explaining how risky assets are priced.

4 The capital market line (CML)

Consider first the portfolio opportunity set available to investors. The next figure illustrates the portfolio opportunity set graphically in expected return - standard deviation space.



In the absence of a riskless asset, investors can choose any portfolio on the boundary, or interior, of KTMJ. The boundary is called the *minimum variance set*. Given this choice, investors will limit their search for a portfolio to those on the upward sloping section of the minimum variance set, TMJ. We call this the *efficient set*. The efficient set is the set of portfolios that offer the highest level of expected return for a given level of risk (standard deviation).

Now assume that the investors also have the opportunity to borrow and lend at the riskless rate r_f. Clearly this assumption is unrealistic for most small investors like you and me. However, large investors can come reasonably close to borrowing at the risk-free interest rate by issuing short-term commercial paper or entering into repurchase agreements with other large investors.

The existence of a riskless asset extends the portfolio opportunities available to investors beyond the efficient set of risky assets. Recall that by combining an riskless asset with a risky asset (or portfolio), an investor can obtain any portfolio lying on a line between the two assets. Thus, by combining the riskless asset with the portfolio M in the figure, the investor can obtain any portfolio on the line labeled CML in the figure. We call this the *Capital Market Line*.

Points between r_f and M represent portfolios in which there is a *long* position in *both* the riskless asset and the risky portfolio M. Points to the right of M represent portfolios in which there is a *long* position in the *risky* portfolio M and a *short* position in the *riskless* asset. For these portfolios, the weight on M is greater than 1 and the weight on the riskless asset is less than zero. As always, these weights must sum to 1.

Question: Why are we not likely to be to the left of the origin?

The expected return and standard deviation of any portfolio falling on the CML are:

$$E[\tilde{r}_p] = \omega r_f + (1 - \omega)E[\tilde{r}_m]$$

$$\sigma_p = (1 - \omega)\sigma_m$$

The equation for the CML is:

$$E[\tilde{r}_p] = r_f + (E[\tilde{r}_m] - r_f)\frac{\sigma_p}{\sigma_m}$$

Notice that under the assumptions of the CAPM, all investors hold portfolios on the CML. Since these portfolios can be created by combinations of the riskless asset and the risky portfolio m, all investors will restrict their portfolio holdings to just these two assets. We call this result the *two-fund separation theorem*. In short, the two-fund separation theorem states that, regardless of investors' risk preferences, they will choose portfolios consisting of only two assets (or mutual funds). These assets are the riskless asset and the risky portfolio m.

We now argue that the portfolio m, in equilibrium, must be the *market portfolio* consisting of all risky assets in proportion to their relative market values.

The reason is that if M was not the market portfolio, then some assets would have excess demand and others would have excess supply. For example, suppose portfolio M included too much of IBM and too little of Exxon (relative to their market values). Then, since all investors hold portfolio m in combination with the riskless asset, there would be excess demand for IBM and excess supply of EXXON. This would cause the price of Exxon to fall and the price of IBM to rise. These price changes will increase demand for Exxon and decrease demand for IBM. In equilibrium, all assets would be held in proportion to their relative market values so that demand and supply are equal. The portfolio weights for assets in the market portfolio are:

$$\omega_j = \frac{\text{Market value of asset } j}{\text{Market value of all assets}}$$

5 Measuring the risk of an individual asset.

The relevant measure of risk for an individual asset is *not* the risk of that asset held in isolation, but is the contribution of that asset to the risk of your portfolio. Since all investors hold the market portfolio under the CAPM, a security's risk can be measured on the basis of its contribution to the variance of the market portfolio. The variance of the market portfolio can be written as follows:

$$\sigma_m^2 = \sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i\sigma_{ij} = \sum_{j=1}^{N}\omega_j\sigma_{jm}$$

Where ω_j is the weight on asset j in the market portfolio m and σ_{jm} is the covariance between asset j and the market portfolio. Thus, the contribution that an individual asset makes to the variance of the market portfolio is measured by its covariance with the market, σ_{jm}.

Now recall the equation for the *beta* of a security.

$$\beta_j = \frac{\sigma_{jm}}{\sigma_m^2}$$

Since the variance of the market, σ_m^2, is the same for all securities, we can measure the risk of a security by either its covariance, σ_{im}, or its beta, β_j. Since beta is more intuitive, we shall use it as a measure of a security's risk.

The beta of the market portfolio is 1:

$$\beta_m = \sum_{j=1}^{N}\omega_j\beta_j = \sum_{j=1}^{N}\omega_j\frac{\sigma_{jm}}{\sigma_m^2} = 1$$

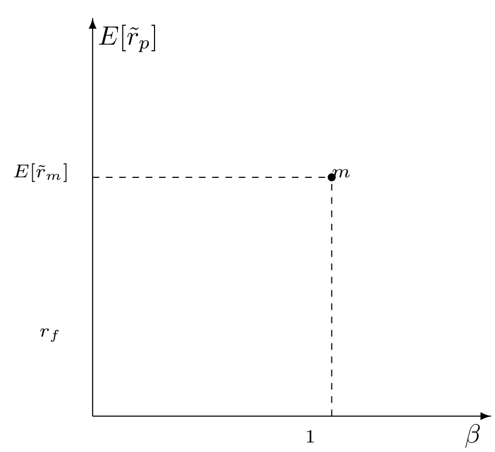
The beta of a portfolio consisting of a fraction ω of the riskless asset and a fraction $(1-\omega)$ of the market portfolio is:

$$\beta_p = \omega \cdot 0 + (1-\omega)\cdot 1.0 = (1-\omega)$$

6 The relationship between risk and expected return.

If beta is the appropriate measure of risk for an individual security, what is the relationship between beta and expected return? Intuitively, securities with higher betas have more systematic (market) risk and, therefore, should command higher expected rates of return in equilibrium.

The figure below plots the relative positions in the riskfree asset ($\beta = 0$) and the market portfolio ($\beta = 1$) in expected return - beta space.

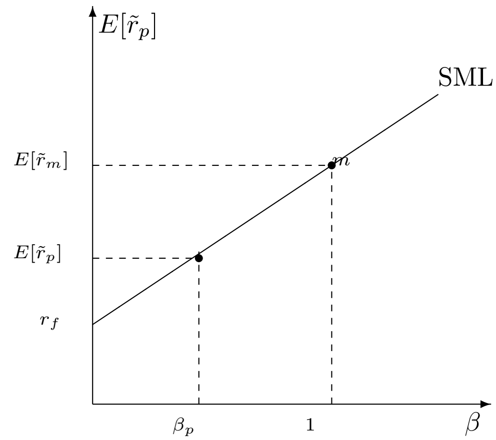


Suppose we combined the risk-free asset with the market portfolio. Where would these portfolios plot in expected return - beta space?

$$E[\tilde{r}_p] = \omega r_f + (1-\omega)E[\tilde{r}_m]$$

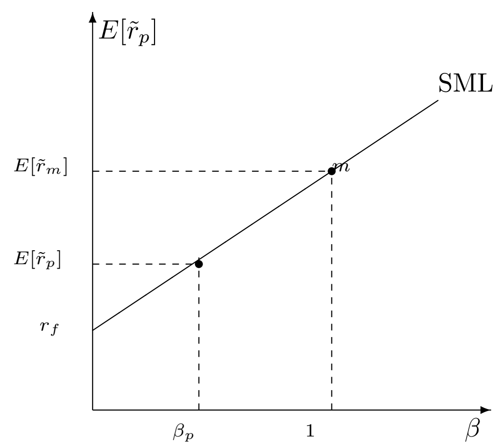
and

$$\beta_p = (1 - \omega)$$



So we see that combinations of the riskfree asset and the market portfolio must fall along the line labelled SML in the previous figure. We call this the *Security Market Line*.

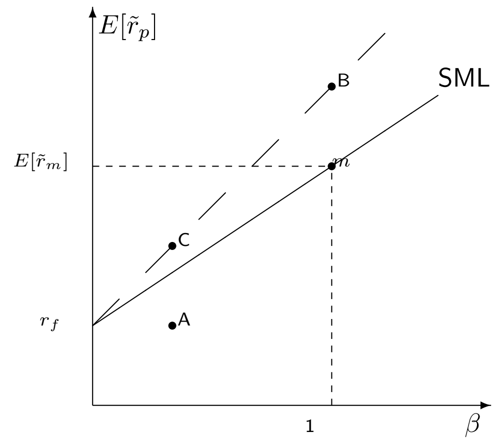
But what about an arbitrary asset or portfolio? Where do they plot in expected return - beta space? Well, to get markets to clear (i.e. demand equals supply), they too must plot along the SML.



To illustrate, consider the next exercise

Exercise 1.

Consider the following illustration of the expected return – beta relation for two stocks, A and B,



Note that the two stocks do not plot on the SML. Stock A plots below and stock B plots above the SML.

1. How would you combine the two in a portfolio?
2. What would be the effects on stock prices?

Solution to Exercise 1.

Since the average beta is 1, if there exist a stock that plots *below* the SML, there must also exist at least one stock that plots *above* the SML. Now ask yourself whether you would hold stock A at all. You would do better if you instead dropped stock A from your portfolio and replaced it with asset C (a combination of the riskfree asset and stock B). By including asset C in your portfolio, you could increase the expected return on your portfolio without increasing the risk. This would create excess demand for stock B and excess supply for stock A. Consequently, in equilibrium, all assets and portfolios must plot on the SML.

The equation of the SML is

$$E[\tilde{r}_j] = r_f + (E[\tilde{r}_m] - r_f)\beta_j$$

where $(E[\tilde{r}_m] - r_f)$ is called the *market risk premium.*

Exercise 2.

You are given the following information

Portfolio	Average annual Rate of return (nominal)	Average annual Rate of return (real)	Average risk premium (Extra return versus T-bills)
Common stocks	12.0	8.8	8.4
Corporate bonds	5.1	2.1	1.7
Government bonds	4.4	1.4	1.0
Treasury bills	3.5	0.4	0.0

Average rates of return for the period 1926 to 1985.

Stock	Beta
AT&T	0.81
Digital Eq.	1.21
Bristol Myers	0.91
Exxon	0.71
General Mills	0.57
MCI Comm.	1.52
Compaq	1.73
Genentech	1.95
Mesa Petroleum	0.68
Holly Sugar	0.62

Betas for selected common stocks. 1981-1986.

Use the above information to compute the expected rates of return on these ten stocks. Assume that the current T-bill rate is 7.5%.

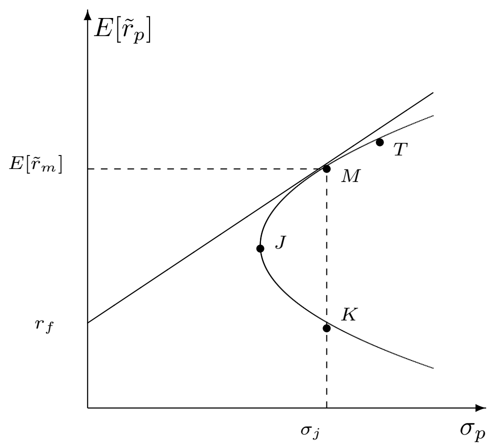
Solution to Exercise 2.

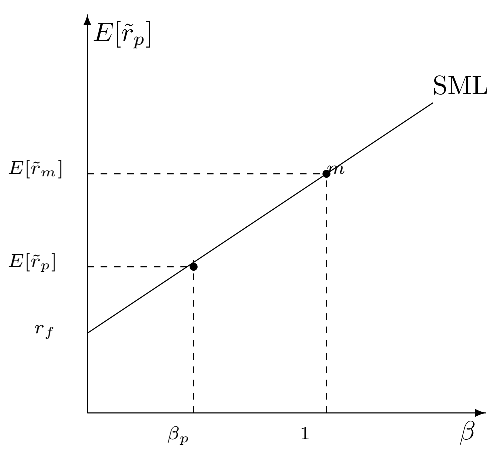
Market risk premium $= (E[\tilde{r}_m] - r_f) = 8.4\%$. Riskfree interest rate $= r_f = 7.5\%$.

Stock	$E[\tilde{r}] = r_f + \beta(E[\tilde{r}_m] - r_f)$
AT&T	14.30
Digital Eq.	17.66
Bristol Myers	15.14
Exxon	13.46
General Mills	12.29
MCI Comm.	20.27
Compaq	22.03
Genentech	23.88
Mesa Petroleum	13.21
Holly Sugar	12.71

7 Relationship between the CML and the SML.

The relationship between the CML and the SML is depicted graphically in the next figure.





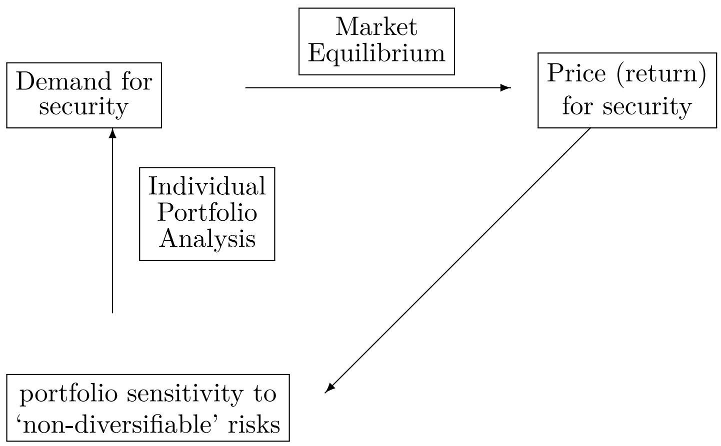
The following points should be obvious by now.

1. All stocks and portfolios plot along the SML, only efficient portfolios plot along the CML.
2. All efficient portfolios plotting on the CML can be obtained by combining the riskfree asset with the market portfolio.
3. All efficient portfolios plotting on the CML are perfectly positively correlated with the market portfolio.

4. All stocks (and other risky assets) lie on the interior of the portfolios opportunity set. They do *not* plot on the CML.

5. Betas combine linearly and measure the systematic (market) risk of an asset. The beta of the market portfolio is 1.0.

8 Basic lesson of the CAPM.



9 Some examples, CAPM usage

10 CAPM and combinations of assets (portfolios)

One often want to look at the riskiness/cost of capital of *portfolios*, combinations of assets/projects.

A very useful feature of CAPM is that it is linear in betas. So suppose we have a collection of projects, indexed by i, with corresponding betas β_i, we can find the riskiness of the portfolio, or the beta of the portfolio β_p, as

$$\beta_p = \sum_i \omega_i \beta_i$$

One could alternatively look at the returns directly, by using the CAPM to go from the beta to return for each individual asset, and then use the individual asset returns to calculate portfolio returns:

$$E[r_i] = r_f + \beta_i(E[r_m] - r_f)$$

$$E[r_p] = \sum \omega_i E[r_i]$$

Exercise 3.

You are given the following information about your portfolio

Stock	beta β_i	weight ω_i
A	1.25	0.25
B	1.00	0.50
C	0.75	0.25

Let $E[\tilde{r}_m] = 10\%$ and $r_f = 5\%$.

1. How do we calculate the expected return on this portfolio?

Solution to Exercise 3.

- Calculate expected returns for each stock using individual betas, and then find the portfolio returns.

$$\begin{aligned} E[\tilde{r}_i] &= r_f + (E[r_m] - r_f)\beta_i \\ E[\tilde{r}_A] &= 0.05 + (0.10 - 0.05)1.25 \\ &= 11.25\% \\ E[\tilde{r}_B] &= 0.05 + (0.10 - 0.05)1.0 \\ &= 10\% \\ E[\tilde{r}_C] &= 0.05 + (0.10 - 0.05)0.75 \\ &= 8.75\% \end{aligned}$$

Expected portfolio return

$$\begin{aligned} E[\tilde{r}_p] &= \sum_{i=1}^{3} \omega_i E[\tilde{r}_i] \\ &= \tfrac{1}{4}11.25\% + \tfrac{1}{2}10\% + \tfrac{1}{4}8.75\% \\ &= 10\% \end{aligned}$$

- Find the beta for the portfolio, and then use this beta to find expected portfolio return.

$$\begin{aligned} \beta_p &= \sum_{i=1}^{3} \omega_i \beta_i \\ &= 0.25 \cdot 1.25 + 0.5 \cdot 1.0 + 0.25 \cdot 0.75 \\ &= 1 \end{aligned}$$

Expected portfolio return

$$E[\tilde{r}_p] = r_f + (E[r_m] - r_f)\beta_p = 10\%$$

10.1 Identifying underpriced stocks

Exercise 4.

Suppose you are the manager of an investment fund in a CAPM world. Ignore taxes. Given the following forecast:

$$E[\tilde{r}_m] = 16\%$$
$$\sigma(r_m) = 0.20$$
$$r_f = 8\%$$

1. Would you recommend investment in a security j with the following characteristics: $E[\tilde{r}_j] = 12\%$ and $\text{cov}(\tilde{r}_j, \tilde{r}_m) = 0.01$?
2. Suppose next period it turns out that this security j has had a return of only 5%. How would you explain this, given that $E[\tilde{r}_j] = 12\%$?

Solution to Exercise 4.

1.

$$\beta_j = \frac{\text{cov}(\tilde{r}_j, \tilde{r}_m)}{\text{var}(\tilde{r}_m)} = \frac{0.01}{(0.20)^2} = \frac{0.01}{0.04} = 0.25$$

$$\begin{aligned} r_j &= r_f + (E[r_m] - r_f)\beta_j \\ &= 0.08 + (0.16 - 0.08)0.25 = 0.10 = 10\% \end{aligned}$$

Since the expected return of 12% is higher than the required return, should invest.

2. The after – the – fact returns may be different from the expected returns. You might just had a bad outcome, the CAPM might still be true.

11 Performance evaluation with CAPM

To gather some intuition for how to think about the CAPM, let us use it to look at the "performance" of a stock investment.

Essentially, it allows us to see whether the return from owning a stock is the same as the risk-compensated required return from owning the stock.

Exercise 5.

A stock has a beta of 0.9. A security analyst who specializes in studying this stock expects its return to be 13%. Suppose the risk free rate is 8% and the market risk premium is 6%.

1. Is the analyst pessimistic or optimistic about this stock relative to the markets expectation?

Solution to Exercise 5.

1. The market expects

$$
\begin{aligned}
E[r] &= r_f + (E[r_m] - r_f)\beta \\
&= 0.08 + 0.06 \cdot 0.9 \\
&= 13.4\%
\end{aligned}
$$

The analyst is pessimistic, since his expectation of 13% is lower than the 13.4% expected return for a stock with $\beta = 0.9$.

12 Use of betas in practice

Finding a relevant beta is a nontrivial exercise. For listed companies, it is possible to use historical data to estimate beta, but even that has issues.

1. Betas may change over time.

   [-] Due to estimation errors.

   [-] Due to changes in the underlying risk characteristics of the firm.

2. Determining the appropriate time period to do the estimation.

If you want to estimate the beta for a new investment project, you would have no past information about how the returns on the new project covary with the market.

12.1 Estimating betas from market data

A typical way of estimating betas is to use stock market data, and estimate the implied beta from historical returns data.

Exercise 6.

You want to estimate the beta of the company Norsk Hydro (NHY), traded at the Oslo Stock Exchange. Use the observations of the stock and the market (an Oslo Stock Exchange Index) in the period 2007–2012.

1. Use daily returns to estimate the beta.

2. Do you get a different estimate if you use monthly returns?

Solution to Exercise 6.

The first step in this exercise is to gather returns data necessary to do the calculations.

This example uses data downloaded from the Oslo Stock Exchange, five years worth of daily prices dnding in 2012. Note that the stock prices are adjusted for corporate events, such as the SEO in july of 2010.

The beta calculation is to find

$$\beta_{NHY} = \frac{\text{cov}(r_m, r_{NHY})}{\text{var}(r_m)}$$

where r_m is the return on the index, and r_{NHY} is the return on NHY.

We therefore first transform prices to returns using

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}}$$

and then use the excel functions for covariance (COVAR) and variance (VAR) to do the calculation

Doing so we find the beta of NHY as

$$\beta_{NHY} = \frac{\text{cov}(r_{NHY}, r_m)}{\text{var}(r_m)} \approx = 1.21$$

This calculation is illustrated in the spreadsheet `nhy_beta.xls` (available on the course page).

To implement the second part of the question, do the calculation from monthly returns, one need to first find monthly returns. While this can be done in a spreadsheet, it is by no means straightforward to pick end-of-month prices. It is much easier to do these calculations as regressions in R, as illustrated next.

Note that the data for doing this is, as well as the code for the R examples, is available on the course homepage

First, using daily returns, running the regression.

Reading the data

```
# read files pulled from OSE homepage
library(zoo)
OSEBXPric <- read.zoo("../data/osebx.csv",
                   header=TRUE,format="%d.%m.%y",sep=",",skip=1)
# pick first column, the closing value
OSEBXPric <- OSEBXPric[,1]
names(OSEBXPric)[1] <- "OSEBXClose"
NHYPric   <- read.zoo("../data/nhy.csv",
                   header=TRUE,format="%d.%m.%y",sep="\t",skip=1)
NHYPric   <- NHYPric[,1]
names(NHYPric)[1] <- "NHYClose"
```

Running the regression on daily data:

```
> OSEBXdRet <- (OSEBXPric-lag(OSEBXPric))/lag(OSEBXPric)
> NHYdRet <- (NHYPric - lag(NHYPric))/lag(NHYPric)
> beta <- cov(OSEBXdRet,NHYdRet) / var(OSEBXdRet)
> print(beta)
[1] 1.216354
> lm(NHYdRet~OSEBXdRet)

Call:
lm(formula = NHYdRet ~ OSEBXdRet)

Coefficients:
(Intercept)    OSEBXdRet
  0.0008635    1.2163540
```

More detailed output

```
lm(formula = NHYdRet ~ OSEBXdRet)

Residuals:
      Min        1Q    Median        3Q       Max
-0.104060 -0.008742 -0.000554  0.009200  0.171459
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.0008635  0.0005197   1.662   0.0969 .
OSEBXdRet   1.2163540  0.0248864  48.876   <2e-16 ***

Residual standard error: 0.01843 on 1256 degrees of freedom
Multiple R-squared:  0.6554,Adjusted R-squared:  0.6551
F-statistic:  2389 on 1 and 1256 DF,  p-value: < 2.2e-16
```

Then doing the regression on monthly data

```
> # now find monthly returns, first monthly prices
> OSEBXMpric <- OSEBXPric[endpoints(OSEBXPric,on="months")]
> NHYMPric <- NHYPric[endpoints(NHYPric,on="months")]
> OSEBXMrets <- (OSEBXMpric-lag(OSEBXMpric))/lag(OSEBXMpric)
> NHYMrets <- (NHYMPric - lag(NHYMPric))/lag(NHYMPric)
> lm (NHYMrets ~ OSEBXMrets)

Call:
lm(formula = NHYMrets ~ OSEBXMrets)

Coefficients:
(Intercept)   OSEBXMrets
    0.01642      1.28031
```

More detailed output

```
lm(formula = NHYMrets ~ OSEBXMrets)

Residuals:
      Min        1Q    Median        3Q       Max
-0.117934 -0.037916 -0.005473  0.034074  0.118008

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.016415   0.007252   2.263   0.0274 *
OSEBXMrets  1.280306   0.081641  15.682   <2e-16 ***


Residual standard error: 0.05607 on 58 degrees of freedom
Multiple R-squared:  0.8092,Adjusted R-squared:  0.8059
F-statistic: 245.9 on 1 and 58 DF,  p-value: < 2.2e-16
```

Frequency	Beta estimate
Daily	1.2163540
Monthly	1.28031

12.2 Industry betas

An alternative method for estimating the beta for a new project or for a division is to use a beta estimate for the industry to which the project belongs. These industry betas are often more reliable than the estimates for individual firms and should provide a relatively good measure of the project or division's market risk.

Industry	Beta
Electronic components	1.49
Crude petroleum and natural gas	1.07
Retail department stores.	.95
Chemicals	.89
Food	.84
Trucking	.83
Paper and allied products	.82
Airlines	.75
Steel	.66
Railroads	.61
Telephone companies	.50
Electric utilities	.46

13 CAPM Summary

Purpose: Formalize a method to account for the *riskiness* in an investment.
Develop under specifi assumptions.

13.1 Mean-Variance preferences

All investors decide their portfolios with preferences that

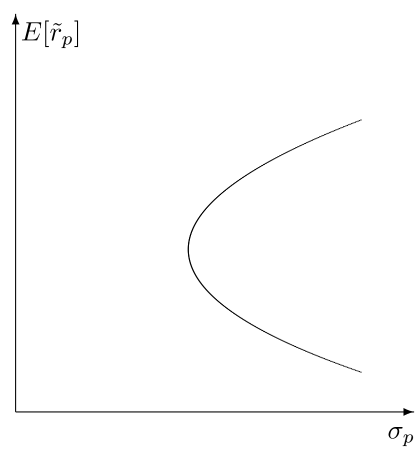
- Prefer higher expected return (mean expected returns)
- Dislike variability of wealth (standard deviation of returns)

A *portfolio* a set of *weights* ω_i for each possible investment asset i.
Portfolio has

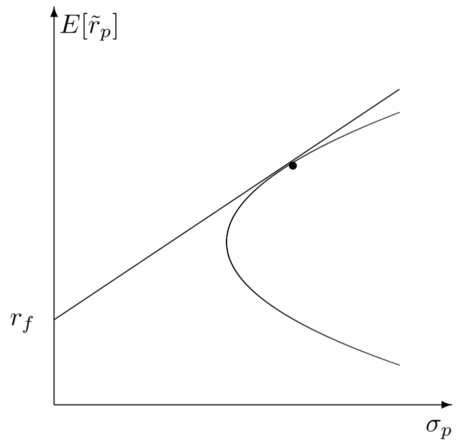
$$\text{Portfolio expected return:} \quad E[r_p] = \sum_{i=1}^{n} \omega_i E[r_i]$$

$$\text{Portfolio variance:} \quad \sigma^2(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \text{cov}(r_i, r_j)$$

Implications (graphically): Possible optimal mean variance combinations wih only risky assets,

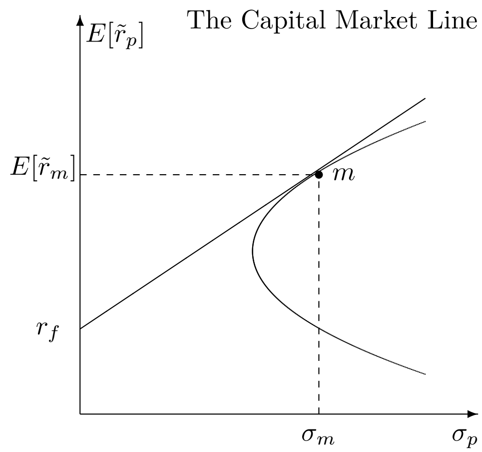


Introduce the possibility of investing/borrowing at the risk free rate r_f, optimal mean/variance portfolios on line between r_f and tangency portfolio on the minimum variance set of risky assets:



13.2 CAPM

Suppose all investors have the same expectations about asset returns, ie. they face the same choice set:



Investors will optimally combine r_f with the portfolio m.
→ Portfolio m is the *market portfolio.*

This is the CAPM.

Implication: For an individual asset, the only risk that matters is the contribution of risk to the market portfolio.

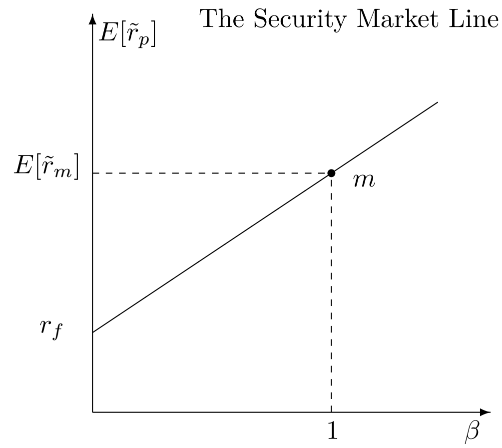
Hence, for an individual asset, its risk is summarized in the assets *beta*:

$$\beta_i = \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)}$$

Using the CAPM is then a matter of estimating beta.

Calculation of expected return given beta

$$E[r_i] = r_f + \beta_i \left(E[r_m] - r_f \right)$$



13.3 Portfolios

Linear in both returns and beta.

ω_i fraction of wealth in asset i.

Portfolio beta

$$\beta_p = \sum_i \omega_i \beta_i$$

$$E[r_p] = r_f + \beta_p(E[r_m] - r_f)$$

Portfolio returns

$$E[r_i] = r_f + \beta_i(E[r_m] - r_f)$$

$$E[r_p] = \sum \omega_i E[r_i]$$

13.4 Corporate implication

The CAPM gives the expected return per unit of risk.

This is also the *required return* for corporate investments with that risk.

14 Notation

$\sigma(r_i)$ standard deviation of return on asset i.
$\sigma^2(r_i)$ variance of return on asset i.
$\rho(r_i, r_j)$ correlation between returns of assets i and j.
$\beta_i = \frac{mboxcov(r_i, r_m)}{\text{var}(r_m)} - -$ Beta
r_D cost of debt capital
r_E cost od equity capital
D market value of debt
E market value of equity
ω_i weight of asset i in portfolio

References

Jonathan Berk and Peter DeMarzo. *Corporate Finance*. Pearson, fifth edition, 2020.