Bond Pricing

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Introduction

Overview of lecture

- Defining bonds
  - Classifying
- Pricing bonds
- Bond Yield to Maturity
- Interest Rate Sensitivity - Bond Duration

Bond - define

A bond, which is the classical example of a *fixed income security*, is a (traded) asset with a predetermined stream of future cash payments.
Each period until the bond matures you are promised an interest payment and at maturity you are also promised the *face value* of the bond.
Distinguishing feature of bonds: The ex ante contractability of the cash flows coming from the security.
Cash flows can be unconditional:
Example: Fixed interest bond issued by government
Cash flows can be conditional:
Example: Floating interest rate bond,

Classifying bonds

Relevant dimensions:

- Issuers
- Maturity
- Contractual features.

Issuers

- Treasury securities (Developed countries). Norges Bank, sertifikater, obligasjoner. US Treasury securities (bills, bonds), UK gilts, etc.
- Agency securities - governmental or other official agencies, or issuers with (implicit) governmental guarantees.
- Corporate Securities. Divided based on creditworthiness of issuing corporation. (Investment grade, noninvestment grade, junk bonds (high yield bonds).
- Mortgage backed securities. Pool large number of individual mortgages into a pool, security issued backed by the pool. (Securitization)
- Asset backed securities. Similar to Mortgage backs, other assets enter into the pool, such as credit card receivables. (Securitization)
- Municipal issues. Local government. In the US important because of their tax status.
- Emerging market securities. e.g. Russian, Mexican govermental debt.

Maturity

- Short term (up to a year) (US: treasury bills) typically no coupon.
- Medium term (1-10 years) (US: treasury notes) coupon bonds
- Long term (above 10 years) (US: treasury bonds)

Contractual Features

- Callability
  - By issuer
  - By buyer
- Convertibility
  - Into some other asset, typically equity of the same company
- How fixed are the payments?
  - Fixed interest payments
  - Floating rate payments
  - Contingent payments.

Key Players

Issuers of Debt Securities

- Governments and their agencies
- Corporations
- Commercial Banks
- States and municipalities
- Special purpose vehicles
- Foreign institutions

Financial Intermediaries

- Primary dealers
- Other dealers
- Investment banks
- Credit rating agencies
- Credit and liquidity enhancers

Institutional and Retail investors

- Governments
- Pension funds
- Insurance companies
- Mutual funds
- Commercial banks
- Foreign institutions

Framework for pricing

Framework for pricing fixed income securities

- Inflation
- Interest risk
- Credit risk
- Liquidity risk
- Timing risk
- Tax
- Foreign exchange risk
- Political risk

Credit risk

There is some risk that the cash flows will not be the contractual ones.
Example: Corporate bond, corporation defaults and enter bankruptcy, in the end 20% of original face value is paid.
Affects the cash flows coming from the fixed income security.

Interest risk

The *value* of the future cash flows changes. Primarily due to interest rate changes.
Example: Buy a treasury security, fixed future coupon payments. Inflation moves from 1% to 50%. Although the cash flows from the bond are the same, their future value goes down.
Change in value of future cash flows reflected in current interest rates.
Interest rates reflect the marginal valuation of one unit of future cash flow.

Liquidity risk

If a security is illiquid, difficult to sell at current prices.
Liquidity: How much must prices move to sell a given quantity of an asset. Highly liquid: low movement. Low liquidity.
Liquidity affects the value of a security, the price at which one can realize the assets one view of what teh security is worth.

Timing risk

When do we get the cash flows from a security?
Example: A callable bond: A bond issuer can choose to retire the bond by paying back the principal early. This possibility will affect the value.

Additional issues

Tax issues
The more complex the tax rules, the more scope for tax options, optimal tax trading and the like.
FX risk. If coupons are in foreign currency, risk that the exchange rate changes, increasing or lowering the value of cash flows in home currency.
Political risk. Typically for foreign, less developed cases.

Pricing summarized

Complex issues
still – pricing fixed income simpler than the usual finance issues, such as pricing of stock.
The analysis of cash flows limited to asking:

1. What is the risk that
   - the cash flows are paid earlier?
   - or are less than the contractual cash flows?
2. What is the cash flows worth, when paid?
   – which comes down to an analysis of the term structure of interest rates, be it risk free or risky.

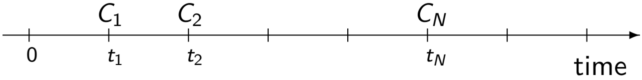
The set of *relevant factors* for pricing are thus fewer and easier to analyze in order to find the exact cash flows consequences.
Flip side: Fixed income analysis need more attention to fine detail.

Bond Pricing

Pricing of fixed income securities is conceptually simple, present value calculation

The present value is the current value of a stream of future payments.

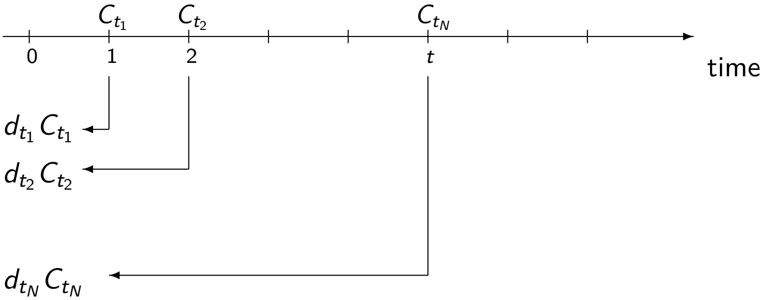
We have N future cash flows C_t that occur at times $t_1, t_2, \cdots, t_N$.



To find the *present* value of these future cash flows one need a set of prices of future cash flows.

Suppose d_t is the price one would pay today for the right to recive one dollar at a future date t.

$$PV = \sum_{i=1}^{N} d_{t_i} C_{t_i}$$



The current *price* d_t for a future payment of \$1 at date t is affected by both the time between now and t (maturity) and the *riskiness* of the future cash flow.

Pricing a bond

Cash flows of the typical bond

$t =$	0	1	$\cdots$	$T-1$	T	$T+1$
Coupon	C_1	C_2	$\cdots$	C_{T-1}	C_T	–
Principal	–	–	$\cdots$	–	F_T	–
Sum	C_1	C_2	$\cdots$	C_{T-1}	$C_T + F_T$	–

The current price of a bond that matures T periods from today is

$$B_0 = \sum_{t=1}^{T} \frac{E[C_t]}{(1+r)^t} + \frac{E[F_T]}{(1+r)^T}$$

where

- B_0 The current bond price.
- $E[C_t] =$ The expected coupon payment in period t.
- $E[F_T] =$ The expected face value payment in period T.
- r the relevant cost of capital

Pricing the most standard bond - fixed coupon, terminal payment of principal

Special case: Fixed coupon C

$t =$	0	1	$\cdots$	$T-1$	T	$T+1$
Coupon	C	C	$\cdots$	C	C	$-$
Principal	$-$	$-$	$\cdots$	$-$	F_T	$-$
Sum	C	C	$\cdots$	C	$C + F_T$	$-$

If the expected coupon payments are the same each period we can value the first term of the above formula as a T-period annuity. That is,

$$B_0 = E[C]\left(\frac{1}{r} - \frac{1}{r(1+r)^T}\right) + \frac{E[F_T]}{(1+r)^T}$$

Exercise

A Treasury bond has a coupon rate of 9%, a face value of \$1000 and matures 10 years from today. For a treasury bond the interest on the bond is paid in semi-annual installments. The current riskless interest rate is 12% (compounded semi-annually).

1. Suppose you purchase the Treasury bond described above and immediately thereafter the riskless interest rate falls to 8%. (compounded semi-annually). What would be the new market price of the bond?
2. What is your best estimate of what the price would be if the riskless interest rate was 9% (compounded semi-annually)?

Exercise Solution

1. If the interest rate is 8%:

$$P_0 = \$45 \left[\frac{1}{0.04} - \frac{1}{0.04 \cdot 1.04^{20}} \right] + \frac{\$1000}{(1.04)^{20}} = \$1067.95$$

2. If the interest rate is 9%: A quick calculation will verify that it is $P_0 = 1000.0$.

$$P_0 = \$45 \left[\frac{1}{0.045} - \frac{1}{0.045 \cdot 1.045^{20}} \right] + \frac{\$1000}{(1.045)^{20}} = \$1000$$

Yield to maturity

Often an investor will know the current market price of a bond and will want to know what rate of return he/she can expect to earn if the bond is purchased.
Measure of return quoted by financial services: *yield to maturity (YTM)* (or the *Internal rate of return (IRR)*).
The YTM is the actual rate of return you will earn by purchasing the bond if

1. You hold the bond until maturity.
2. The bond is not called and does not default.

Yield to Maturity

The YTM is the rate of interest that equates the present value of the *promised* interest and face value payments with the current market price.

YTM is the interest rate y that solves

$$P_0 = C\left(\frac{1}{y} - \frac{1}{y(1+y)^T}\right) + \frac{F_T}{(1+y)^T}$$

Note, however, that the YTM *overestimates* the investors expected rate of return from holding the bond until maturity if there is a possibility of default.

Exercise

A \$100, 10 year bond was issued 7 years ago at a 10% annual interest rate. The current interest rate is 9%. The current price of the bond is 100.917. Use annual, discrete compounding.

1. Calculate the bonds yield to maturity.

Exercise Solution

1. YTM: Calculate the internal rate of return on:

t	$=$	0	1	2	3
C_t	$=$	-100.917	10	10	110

$\text{IRR} = 0.096344 = 9.6344\%$

Interest rate sensitivity

The market price of a bond is inversely related to the market rate of interest and is equal to the face value of the bond only when the market rate of interest equals the coupon rate. That is

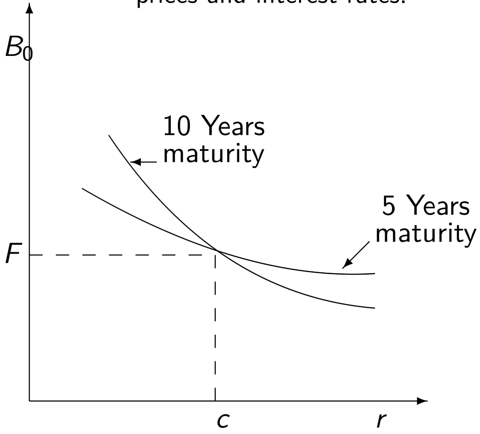
$$B_0 > F \text{ if } r < c$$

$$B_0 = F \text{ if } r = c$$

$$B_0 < F \text{ if } r > c$$

Interest rate sensitivity

Relationship between bond prices and interest rates.



Interest rate sensitivity

Bonds with longer maturities are more sensitive to interest rate changes than bonds with shorter maturities.
In diagram: the curve is steeper for bonds with longer maturities.
The slope of the curve is

$$
\begin{aligned}
&\text{Slope} \\
&= -\frac{1}{1+r}\left[\sum_{t=1}^{T} \frac{t \cdot C_t}{(1+r)^t} + \frac{T \cdot F_T}{(1+r)^T}\right] \\
&= -\frac{1}{1+r}\left[\sum_{t=1}^{T} t \cdot PV(C_t) + T \cdot PV(F_T)\right]
\end{aligned}
$$

Interpretation of slope:
How much the price of the bond will decline with a one unit rise in interest rates.

Interest rate sensitivity

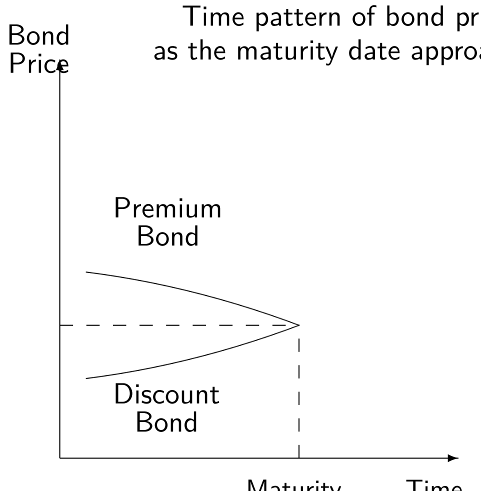
The slope – how much the price of the bond will decline with a one unit rise in interest rates.
To determine the *percentage change* in the price of the bond, divide the slope by the initial price of the bond, P_0:

$$\begin{aligned}&\text{Percentage change in bond price}\\ &= \quad -\frac{1}{1+r}\left[\sum_{t=1}^{T}\frac{t\cdot PV(C_t)}{P_0}+\frac{T\cdot PV(F_T)}{P_0}\right]\end{aligned}$$

The expression in brackets is called the *duration* of the bond.
Duration is interpreted as the *weighted average maturity.*

Interest rate sensitivity depend on time to maturity

Bond Price sensitivity to interest rate changes, as for example measured by duration, will be less and less as the bond approaches maturity.



Bond Duration

Recall what we found

$$\text{Percentage change in bond price} = -\frac{1}{1+r}\left[\sum_{t=1}^{T}\frac{t \cdot PV(C_t)}{P_0} + \frac{T \cdot PV(F_T)}{P_0}\right]$$

The expression in brackets is called the *duration* of the bond. It can be interpreted as the *weighted average maturity.*

Exercise

Suppose you are trying to determine the interest rate sensitivity of two bonds. Bond 1 is a 12% coupon bond with a 7-year maturity and a \$1000 principal. Bond 2 is a 'zero-coupon' bond that pays \$1000 after 7 years. The current interest rate is 12%.

1. Determine the duration of each bond.
2. If the interest rate increases 100 basis points (100 basis points = 1%), what will be the capital loss on each bond?

Solution

Duration

	Cash Flow		PV(r=12%)	
Year	Bond 1	Bond 2	Bond 1	Bond 2
1	120	0	107.14	-
2	120	0	95.66	-
3	120	0	85.41	-
4	120	0	76.26	-
5	120	0	68.09	-
6	120	0	60.80	-
7	120	0	54.28	-
7	1000	1000	452.34	452.34
P_0			1000.00	452.34

Solution ctd

Duration bond 1:

$$(107.14 + 95.66 \cdot 2 + 85.41 \cdot 3 + 76.26 \cdot 4 + 68.09 \cdot 5$$

$$+60.80 \cdot 6 + 507.63 \cdot 7)/1000 = 5.11139$$

Duration bond 2:

$$\frac{452.34 \cdot 7}{452.34} = 7$$

Bond 2 will be more sensitive to interest rate changes.

Solution ctd

If the interest rate increases 100 basis points to 13%, the new prices of each bond will be:

$$\text{Price Bond 1} = \sum_{t=1}^{T} \frac{\$120}{(1.13)^t} + \frac{\$1000}{(1.13)^7} = \$955.77$$

$$\text{Price Bond 2} = \frac{1000}{1.13^7} = 425.06$$

$$\text{Capital Loss Bond 1} = 1000 - 955.77 = 44.23$$

$$\text{Percentage Loss Bond 1} = \frac{44.23}{1000} = 4.423\%$$

$$\text{Capital Loss Bond 2} = 452.34 - 425.06 = 27.28$$

$$\text{Percentage Loss Bond 2} = \frac{27.28}{452.34} = 6.03\%$$

Note: The percentage loss on each bond is approximately equal to

$$\text{Percentage Loss} \approx \frac{\text{Duration}}{1+r} \cdot \Delta r$$

$$\text{Percentage Loss Bond 1} \approx \frac{5.11139}{1.12} \cdot 0.01 = 4.563\%$$

Convexity

Duration $\rightarrow$ *Slope* of change in price as interest rate changes.
In mathematics: First derivative.

$$\frac{\Delta P}{P} \approx -D^{*} \Delta y$$

Also from mathematics: First order derivative is only an approximation, to be more accurate also account for second order effects.
Convexity: The *curvature* of the relationship between bond prices and interest rates.

Convexity

Calculate convexity of a bond with T periods left as:

$$\text{Convexity} = \frac{1}{P(1+r)^2} \sum_{t=1}^{T} (t + t^2) PV(C_t)$$

where C_t is the cash flow at time t and $PV(\cdot)$ is the present value operator.
Modify approximation of the change in price using both first and second order derivatives:

$$\frac{\Delta P}{P} = -D^* \Delta y + \frac{1}{2} \times \text{Convexity} \times (\Delta y)^2$$

Exercise - Bond Pricing and Interest Rate Sensitivity

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Find the bond's current price.
2. Suppose the interest rate changes to 10%, determine the new price of the bond by direct calculation.
3. Instead of direct calculation, use duration to estimate the new price and compare it to the correct price.
4. Use convexity to improve on your estimation using duration.

Exercise - Bond Pricing and Interest Rate Sensitivity - solution

The bond price:

t	$=$	0	1	2	3
C_t	$=$	0	10	10	110

$$NPV = 0 + \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531$$

If the interest rate increases to 10%, the bond will be selling at par, equal to 100, which can be confirmed with direct computation:

$$NPV = 0 + \frac{10}{(1+0.1)^1} + \frac{10}{(1+0.1)^2} + \frac{110}{(1+0.1)^3} = 100$$

Exercise - Bond Pricing and Interest Rate Sensitivity - solution

Calculate the bond's **duration**:

t	C_t	$PV(C_t)$	$tPV(C_t)$
1	10	9.2	9.2
2	10	8.4	16.8
3	110	84.9	254.8
Sum		102.5	280.8
Bondprice			102.531
Duration			2.74

Modified duration:

$$D^* = \frac{D}{1+r} = \frac{2.74}{1.09} = 2.51$$

Exercise - Bond Pricing and Interest Rate Sensitivity - solution

Let us now calculate the change in the bond price

$$\frac{\Delta P}{P} = -D^{*}\Delta y = -2.51 \cdot 0.01 = -0.0251$$

Which means theat the bond price changes to:

$$P+\delta P = 102.531+\left(\frac{\Delta P}{P}\right)P = 102.531-0.0251\cdot 102.531 = 99.957$$

Calculate the bond's convexity

t	C_t	$PV(C_t)$	$PV(C_t)(t^2+t)$
1	10	9.2	18.3
2	10	8.4	50.5
3	110	84.9	1019.3
Sum		102.5	1088.1
Bondprice			102.531
Convexity			8.93

Exercise - Bond Pricing and Interest Rate Sensitivity - solution

Recalculating the change in the bond price using convexity:

$$\frac{\Delta P}{P} = -D^{*}\Delta y + \frac{1}{2}(\text{Convexity})$$

$$= -2.51 \cdot 0.01 + \frac{1}{2} 8.93 (0.01)^2$$

$$= -0.0251 + 0.00044 = -0.02465$$

Use this to re-estimate the bond price:

$$P + \delta P = 102.531 \left(1 + \left(\frac{\Delta P}{P}\right) P\right) = 102.531(1 - 0.02465) = 100.00$$

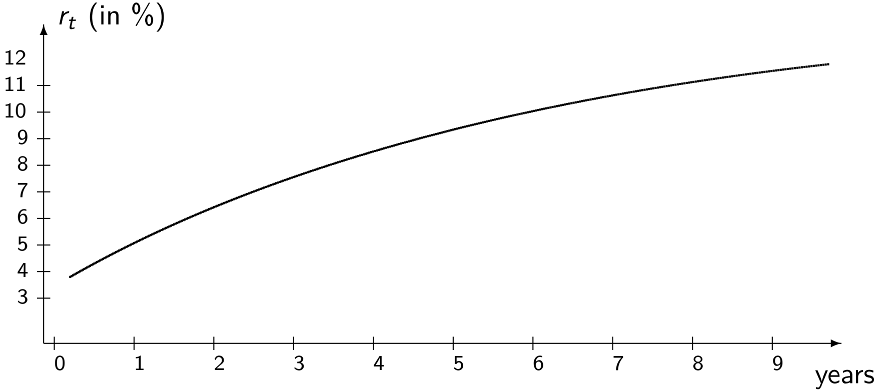
Bond Pricing - term structure of interest rates

The simple bond pricing formula

$$B_0 = E[C]\left(\frac{1}{r} - \frac{1}{r(1+r)^T}\right) + \frac{E[F_T]}{(1+r)^T}$$

is only valid when there is a single interest rate (cost of capital) r.
In practice the world is not that simple.
Interest rates used for disounting varies with the time the payment occurs – the *term structure of interest rates*

Bond Pricing - term structure of interest rates



Bond Pricing - term structure of interest rates

Need to use the term structure for pricing the bond, either

In terms of spot rates (r_t):

$$B_0 = \sum_{t=1}^{T} \frac{E[C_t]}{(1 + r_t)^t} + \frac{E[F_T]}{(1 + r_t)^T}$$

or

In terms of discount factors (d_t):

$$B_0 = \sum_{t=1}^{T} d_t E[C_t] + d_T E[F_T]$$

Exercise

A bond promises the following sequence of payments:

t	$=$	1	2	3	4
Cashflow X_t	$=$	10	10	10	110

The interest rates r_t and prices d_t of future risk free cash flows are as follows

t	$=$	1	2	3	4
r_t	$=$	5.3%	5.4%	5.6%	5.7%
d_t	$=$	0.95	0.9	0.85	0.80

Interest rates are compounded annually.

1. Calculate the bond's price

Exercise solution

$$\text{Bond Price} = \frac{1}{1+r_1}10 + \frac{1}{(1+r_2)^2}10 + \frac{1}{(1+r_3)^3}10 + \frac{1}{(1+r_4)^4} \cdot 110$$

$$\text{Bond Price} = 0.95 \cdot 10 + 0.9 \cdot 10 + 0.85 \cdot 10 + 0.8 \cdot 110$$

$$\text{Bond Price} = 115$$

Bond duration under a nonflat term structure with discrete commpounding

Remember the duration as the sensitivity of the bond price to changes in *the* interest rate in the flat term structure setting, where we calculate duration as:

$$D = \frac{1}{B_0} \sum_{t=1}^{T} tPV(C_t)$$

which in the flat term structure setting is:

$$D = \frac{1}{B_0} \sum_{t=1}^{T} \frac{tC_t}{(1+r)^t}$$

With a nonflat term structure we have to adjust this.
First, what are we using duration for? Still want to measure sensitivity to changes in interest rates, but can no longer look at change in *the* interest rate. Therefore look at sensitivity to *parallell changes* in the level of interest rates.

Bond duration under a nonflat term structure with discrete commpounding

Calculations:
Time t bond price (B_0)

$$B_0 = \sum_i \frac{C_{t_i}}{(1 + r(t, t_i))^{t_i}}$$

Duration (D)

$$D = \frac{1}{B_0} \sum_i t_i PV(C_{t_i})$$

$$D = \frac{1}{B_0} \sum_i t_i d(t, t_i) C_{t_i}$$

$$D = \frac{1}{B_0} \sum_i t_i \frac{C_{t_i}}{(1 + r(t, t_i))^{t_i}}$$

Bond duration under a nonflat term structure with discrete commpounding

An alternative formulation of duration is called the Macaulay duration, and it involves the yield to maturity y of the bond.

$$D = \frac{1}{B_0} \sum_i t_i \frac{C_{t_i}}{(1+y)^{t_i}}$$

Yield to maturity y solves

$$B_0 = \sum_i \frac{C_{t_i}}{(1+y)^{t_i}},$$

hence the above can also be written as

$$D = \frac{\sum_i t_i \frac{C_{t_i}}{(1+y)^{t_i}}}{\sum_i \frac{C_{t_i}}{(1+y)^{t_i}}}$$

When we used duration to measure interest rate sensitivity we also had a second order term involving the bonds *convexity*.

Bond duration under a nonflat term structure with discrete commpounding

Convexity (Cx) is calculated as:
Using the whole term structure

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) PV(C_{t_i})$$

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) d(t, t_i) C_{t_i}$$

With the Macaulay type of calculation using the yield to maturity y in calculating the present value.

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) \frac{C_{t_i}}{(1+y)^{t_i}}$$

Exercise

A 10%, two year bond is traded at a price of 90. The current one year spot rate is $r(0,1) = 12\%$ (with discrete, annual compounding). The bond has a face value of 100.

1. Determine the duration and convexity of the bond, using both the full term structure and the Macaulay style calculations.

Exercise Solution

First need to find the two year spot rate

$$90 = 10d_1 + 110d_2$$

$$d_1 = \frac{1}{1 + 0.12} = 0.89286$$

$$d_2 = \frac{90 - 10d_1}{110} = 0.73701$$

$$r_2 = 0.16483$$

Exercise Solution

Here are some of the calculation in a matrix tool

```
> C=[10 110]
C =
   10  110
> r(1)=0.12
r =   0.12000
> d(1)=1/(1+r(1))
d =   0.89286
> d(2)=(90-10*d(1))/110
d =
  0.89286  0.73701
> d=d'
d =
  0.89286
  0.73701
> BondPrice=C*d
BondPrice = 90
```

Exercise Solution

```
> r(2)=d(2)^(-1/2)-1
r =
  0.12000  0.16483
> y = irr([-BondPrice C],0)
y = 0.16249x
> checkprice=C(1)/(1+y)+C(2)/(1+y)^2
checkprice = 90
```

Exercise Solution

We calculate duration using the two definitions

```
> Duration=1/BondPrice * ( 1*d(1)*C(1) + 2*d(2)*C(2))
Duration = 1.9008
> Duration=1/BondPrice * ( 1*C(1)/(1+y) + 2*C(2)/((1+y)^2)
Duration = 1.9044
```

Using the term structure we find duration as

$$D = 1.9008$$

using the Macaulay definition we find

$$D = 1.9044$$

Thus, not a major difference.

Exercise Solution

We also calculate the convexity for the two definitions

```
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*d(1)*C(1) + (2+2^2)*
Cx = 4.1463
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*C(1)/(1+y) + (2+2^2)
Cx = 4.1570
```

Again, not a major difference with the two methods of calculating