Bond Pricing

Bernt Arne Ødegaard

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1 Introduction

Overview of lecture

- Defining bonds
  - Classifying
- Pricing bonds
- Bond Yield to Maturity
- Interest Rate Sensitivity - Bond Duration

2 What is a Bond?

A bond, which is the classical example of a *fixed income security*, is a (traded) asset with a predetermined stream of future cash payments.

When you purchase a bond, you are acquiring an asset that has a predetermined stream of promised cash payments. Each period until the bond matures you are promised an interest payment and at maturity you are also promised the *face value* of the bond.

Distinguishing feature of bonds: The ex ante contractability of the cash flows coming from the security.

Cash flows can be unconditional:

Example: Fixed interest bond issued by government

Cash flows can be conditional:

Example: Floating interest rate bond,

2.1 Classifying bonds

Relevant dimensions: Issuers, maturity, contractual features.

2.1.1 Issuers

- Treasury securities (Developed countries). Norges Bank, sertifikater, obligasjoner. US Treasury securities (bills, bonds), UK gilts, etc.
- Agency securities - governmental or other official agencies, or issuers with (implicit) governmental guarantees.
- Corporate Securities. Divided based on creditworthiness of issuing corporation. (Investment grade, noninvestment grade, junk bonds (high yield bonds).
- Mortgage backed securities. Pool large number of individual mortgages into a pool, security issued backed by the pool. (Securitization)
- Asset backed securities. Similar to Mortgage backs, other assets enter into the pool, such as credit card receivables. (Securitization)
- Municipal issues. Local government. In the US important because of their tax status.
- Emerging market securities. e.g. Russian, Mexican govermental debt.

2.1.2 Maturity

- Short term (up to a year) (US: treasury bills) typically no coupon.
- Medium term (1-10 years) (US: treasury notes) coupon bonds
- Long term (above 10 years) (US: treasury bonds)

2.1.3 Contractual features

- Callability
  - By issuer
  - By buyer
- Convertibility
  - Into some other asset, typically equity of the same company
- How fixed are the payments?
  - Fixed interest payments
  - Floating rate payments
  - Contingent payments.

2.2 Key players

Issuers of Debt Securities
Governments and their agencies
Corporations
Commercial Banks
States and municipalities
Special purpose vehicles
Foreign institutions
Financial Intermediaries
Primary dealers
Other dealers
Investment banks
Credit rating agencies
Credit and liquidity enhancers
Institutional and Retail investors
Governments
Pension funds
Insurance companies
Mutual funds
Commercial banks
Foreign institutions
Households

A useful framework for thinking about pricing is:

Framework for pricing fixed income securities

- Inflation
- Interest risk
- Credit risk
- Liquidity risk
- Timing risk
- Tax
- Foreign exchange risk
- Political risk

This puts names on the relevant factors and is a checklist for pricing.

Let us look at each of these

Credit risk:

There is some risk that the cash flows will not be the contractual ones.

Example: Corporate bond, corporation defaults and enter bankruptcy, in the end 20% of original face value is paid.

Affects the cash flows coming from the fixed income security.

Interest risk

The *value* of the future cash flows changes. Primarily due to interest rate changes.

Example: Buy a treasury security, fixed future coupon payments. Inflation moves from 1% to 50%. Although the cash flows from the bond are the same, their future value goes down.

This change in the value of future cash flows will be reflected in current interest rates.

Interest rates after all reflect the marginal valuation of one unit of future cash flow.

Liquidity risk

If a security is illiquid, difficult to sell at current prices.

Liquidity: How much must prices move to sell a given quantity of an asset. Highly liquid: low movement. Low liquidity.

Liquidity affects the value of a security, the price at which one can realize the assets one view of what teh security is worth.

Timing risk

When do we get the cash flows from a security?

Example: A callable bond: A bond issuer can choose to retire the bond by paying back the principal early. This possibility will affect the value.

Tax issues

The more complex the tax rules, the more scope for tax options, optimal tax trading and the like.

FX risk. If coupons are in foreign currency, risk that the exchange rate changes, increasing or lowering the value of cash flows in home currency.

Political risk. Typically for foreign, less developed cases.

2.3 Pricing summarized

As the examples show, although complex issues here, still a degree to which the pricing problems analyzed are simpler than the usual finance issues, such as pricing of stock.

The analysis of cash flows limited to asking:

1. What is the risk that
    - the cash flows are paid earlier?

- or are less than the contractual cash flows?

2. What is the cash flows worth, when paid?
   – which comes down to an analysis of the term structure of interest rates, be it risk free or risky.

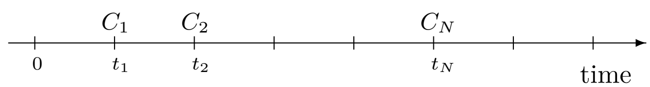
The set of *relevant factors* for pricing are thus fewer and easier to analyze in order to find the exact cash flows consequences.

The flip side to this is that in analysing fixed income problems we have to pay more attention to the fine detail, both in the contracts, and how uncertainty, as reflected in the term structure, evolves.

3 Bond Pricing

Pricing of fixed income securities is conceptually simple.

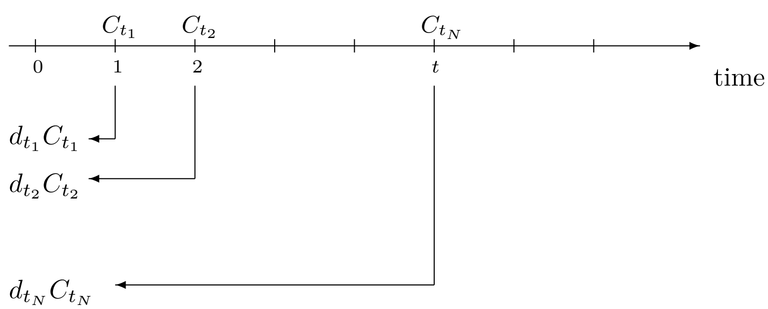
The calculation of present value is one of the basics of finance. The present value is the current value of a stream of future payments. Let C_t be the cash flow at time t. Suppose we have N future cash flows that occur at times $t_1, t_2, \cdots, t_N$.



To find the *present* value of these future cash flows one need a set of prices of future cash flows.

Suppose d_t is the price one would pay today for the right to recive one dollar at a future date t. If one knows this set of prices one would calculate the present value as the sum of the preent values of the different elements.

$$PV = \sum_{i=1}^{N} d_{t_i} C_{t_i}$$



The current *price* d_t for a future payment of \$1 at date t is affected by both the time between now and t (maturity) and the *riskiness* of the future cash flow. Two cash flows at the same future date buth with different risk will have different prices.

To price fixed income securites:
Figure out the relevant cash flows.
Ask: is there some uncertainty about the cash flows?
If so, what is the source of the uncertainty?
Is the uncertainty *contingent* on something? (derivative)

Based on these answers, find the correct prices for future payments
If no contingencies, discount back and find the current price.
If contingent payments, need a more advanced term structure derivatives pricing model.

3.1 Pricing the most standard bond - fixed coupon, terminal payment of principal

Cash flows of the typical bond

$t=$	0	1	$\cdots$	$T-1$	T	$T+1$
Coupon	C_1	C_2	$\cdots$	C_{T-1}	C_T	–
Principal	–	–	$\cdots$	–	F_T	–
Sum	C_1	C_2	$\cdots$	C_{T-1}	C_T+F_T	–

It is equal to the expected stream of payments discounted at the opportunity cost of capital.
The current price of a bond that matures T periods from today is

$$B_0=\sum_{t=1}^{T}\frac{E[C_t]}{(1+r)^t}+\frac{E[F_T]}{(1+r)^T}$$

where

- $E[C_t]=$ The expected coupon payment in period t.
- $E[F_T]=$ The expected face value payment in period T.
- r the relevant cost of capital

Special case: Fixed coupon C

$t=$	0	1	$\cdots$	$T-1$	T	$T+1$
Coupon	C	C	$\cdots$	C	C	–
Principal	–	–	$\cdots$	–	F_T	–
Sum	C	C	$\cdots$	C	$C+F_T$	–

If the expected coupon payments are the same each period we can value the first term of the above formula as a T-period annuity. That is,

$$B_0=E[C]\left(\frac{1}{r}-\frac{1}{r(1+r)^T}\right)+\frac{E[F_T]}{(1+r)^T}$$

Exercise 1.

A Treasury bond has a coupon rate of 9%, a face value of $1000 and matures 10 years from today. For a treasury bond the interest on the bond is paid in semi-annual installments. The current riskless interest rate is 12% (compounded semi-annually).

1. Suppose you purchase the Treasury bond described above and immediately thereafter the riskless interest rate falls to 8%. (compounded semi-annually). What would be the new market price of the bond?
2. What is your best estimate of what the price would be if the riskless interest rate was 9% (compounded semi-annually)?

Solution to Exercise 1.

1. If the interest rate is 8%:

$$P_0=\$45\left[\frac{1}{0.04}-\frac{1}{0.04\cdot 1.04^{20}}\right]+\frac{\$1000}{(1.04)^{20}}=\$1067.95$$

2. If the interest rate is 9%: A quick calculation will verify that it is $P_0=1000.0$.

$$P_0=\$45\left[\frac{1}{0.045}-\frac{1}{0.045\cdot 1.045^{20}}\right]+\frac{\$1000}{(1.045)^{20}}=\$1000$$

3.2 Yield to maturity.

Often an investor will know the current market price of a bond and will want to know what rate of return he/she can expect to earn if the bond is purchased. This, of course, will depend upon a number of factors, including the investor's planned holding period, the default risk of the bond and the expected level of interest rates at the time the bond is going to be sold.

There is however a measure of return that is often used an quoted by financial services called the *yield to maturity (YTM)* or the *Internal rate of return (IRR)*. The YTM is the actual rate of return you will earn by purchasing the bond if

1. You hold the bond until maturity.
2. The bond is not called and does not default.

The YTM is independent of what actually happens to interest rates between now and the bond's maturity date.

The YTM is the rate of interest that equates the present value of the *promised* interest and face value payments with the current market price. That is, the YTM is equal to the interest rate y that solves the following equation

$$P_0 = C\left(\frac{1}{y} - \frac{1}{y(1+y)^T}\right) + \frac{F_T}{(1+y)^T}$$

Note, however, that the YTM *overestimates* the investors expected rate of return from holding the bond until maturity if there is a possibility of default.

Exercise 2.

A \$100, 10 year bond was issued 7 years ago at a 10% annual interest rate. The current interest rate is 9%. The current price of the bond is 100.917. Use annual, discrete compounding.

1. Calculate the bonds yield to maturity.

Solution to Exercise 2.

1. YTM: Calculate the internal rate of return on:

t	$=$	0	1	2	3
C_t	$=$	-100.917	10	10	110

$$\text{IRR} = 0.096344 = 9.6344\%$$

Basically, the only way to solve for the YTM is by trial and error. However, our first guess can always be chosen somewhat strategically. Because the market price of the bond is below its face value, we know that y must be greater than the coupon rate. In this example, the semi-annual coupon rate is $\frac{50}{1000} = 0.05$. Therefore, YTM > 0.05.

Note: Financial calculators and computer spreadsheets have procedures for calculating the IRR.

4 Interest rate sensitivity

The above examples illustrate the following important result.

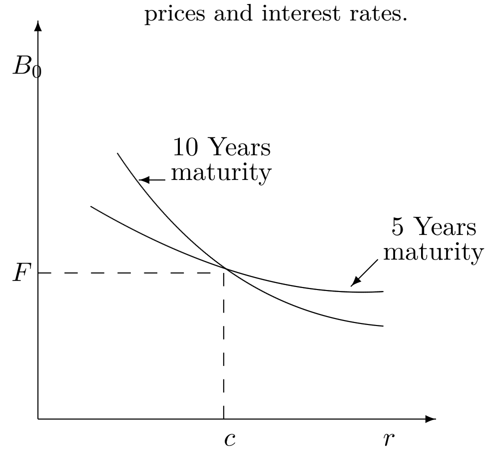
The market price of a bond is inversely related to the market rate of interest and is equal to the face value of the bond only when the market rate of interest equals the coupon rate. That is

$$B_0 > F \text{ if } r < c$$

$$B_0 = F \text{ if } r = c$$

$$B_0 < F \text{ if } r > c$$

Relationship between bond prices and interest rates.



Bonds with longer maturities are more sensitive to interest rate changes than bonds with shorter maturities. In terms of the previous diagram, this means the curve is steeper for bonds with longer maturities. The slope of the curve is

$$\begin{aligned}
&\text{Slope} \\
&= -\frac{1}{1+r}\left[\sum_{t=1}^{T}\frac{t\cdot C_t}{(1+r)^t} + \frac{T\cdot F_T}{(1+r)^T}\right] \\
&= -\frac{1}{1+r}\left[\sum_{t=1}^{T} t\cdot PV(C_t) + T\cdot PV(F_T)\right]
\end{aligned}$$

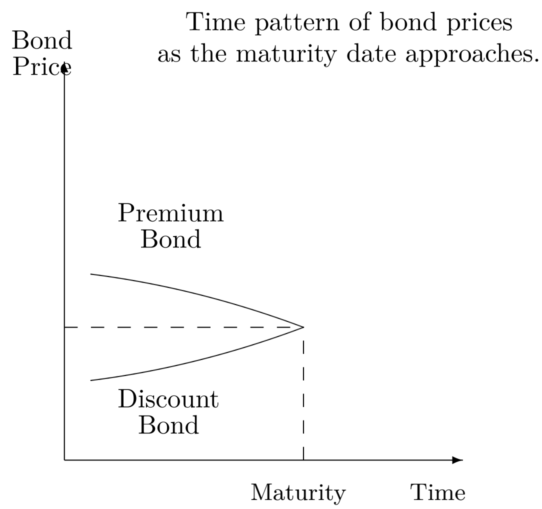
The slope tells us how much the price of the bond will decline with a one unit rise in interest rates. To determine the *percentage change* in the price of the bond, we must divide the slope by the initial price of the bond, P_0. This yields.

$$\begin{aligned}
&\text{Percentage change in bond price} \\
&= -\frac{1}{1+r}\left[\sum_{t=1}^{T}\frac{t\cdot PV(C_t)}{P_0} + \frac{T\cdot PV(F_T)}{P_0}\right]
\end{aligned}$$

The expression in brackets is called the *duration* of the bond. It can be interpreted as the *weighted average maturity.*

4.1 Interest rate sensitivity depend on time to maturity

This sensitivity, as for example measured by duration, will be less and less as the bond approaches maturity, as illustrated by the following figure.



5 Bond Duration

Recall what we found

$$\begin{aligned} &\text{Percentage change in bond price} \\ = \quad & -\frac{1}{1+r}\left[\sum_{t=1}^{T}\frac{t\cdot PV(C_t)}{P_0}+\frac{T\cdot PV(F_T)}{P_0}\right] \end{aligned}$$

The expression in brackets is called the *duration* of the bond. It can be interpreted as the *weighted average maturity.*

Let us first take a look at calculation of duration

Exercise 3.

Suppose you are trying to determine the interest rate sensitivity of two bonds. Bond 1 is a 12% coupon bond with a 7-year maturity and a $1000 principal. Bond 2 is a 'zero-coupon' bond that pays $1000 after 7 years. The current interest rate is 12%.

1. Determine the duration of each bond.
2. If the interest rate increases 100 basis points (100 basis points = 1%), what will be the capital loss on each bond?

Solution to Exercise 3.

1. Duration

	Cash Flow		PV(r=12%)	
Year	Bond 1	Bond 2	Bond 1	Bond 2
1	120	0	107.14	-
2	120	0	95.66	-
3	120	0	85.41	-
4	120	0	76.26	-
5	120	0	68.09	-
6	120	0	60.80	-
7	120	0	54.28	-
7	1000	1000	452.34	452.34
P_0			1000.00	452.34

Duration bond 1:

$$\frac{[107.14+95.66\cdot 2+85.41\cdot 3+76.26\cdot 4+68.09\cdot 5+60.80\cdot 6+507.63\cdot 7]}{1000}=5.11139$$

Duration bond 2:

$$\frac{452.34\cdot 7}{452.34}=7$$

Bond 2 will be more sensitive to interest rate changes.

2. If the interest rate increases 100 basis points to 13%, the new prices of each bond will be:

$$\text{Price Bond 1}=\sum_{t=1}^{T}\frac{\$120}{(1.13)^t}+\frac{\$1000}{(1.13)^7}=\$955.77$$

$$\text{Price Bond 2}=\frac{1000}{1.13^7}=425.06$$

$$\text{Capital Loss Bond 1}=1000-955.77=44.23$$

$$\text{Percentage Loss Bond 1}=\frac{44.23}{1000}=4.423\%$$

$$\text{Capital Loss Bond 2}=452.34-425.06=27.28$$

$$\text{Percentage Loss Bond 2} = \frac{27.28}{452.34} = 6.03\%$$

Note: The percentage loss on each bond is approximately equal to

$$\text{Percentage Loss} \approx \frac{\text{Duration}}{1+r} \cdot \Delta r$$

$$\text{Percentage Loss Bond 1} \approx \frac{5.11139}{1.12} \cdot 0.01 = 4.563\%$$

$$\text{Percentage Loss Bond 2} \approx \frac{7.0}{1.12} \cdot 0.01 = 6.25\%$$

For those who like to see this done formally

Exercise 4.

Consider the pricing of a bond with a flat term structure, where C_t is the cash flow in period t.

$$P_0 = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = \sum_{t=1}^{\infty} C_t \left(\frac{1}{1+r}\right)^t$$

1. Determine the first derivative of the price with respect to the interest rate.
2. Find the duration part of this expression.

Solution to Exercise 4.

1. $$\frac{dP_0}{dr} = \sum_{t=1}^{\infty} C_t t \left(\frac{1}{1+r}\right)^{t-1} \left(\frac{-1}{(1+r)^2}\right) = -\frac{1}{1+r} \sum_{t=1}^{\infty} C_t t \left(\frac{1}{1+r}\right)^t = -\frac{1}{1+r} \sum_{t=1}^{\infty} \frac{tC_t}{(1+r)^t}$$
2. Duration
$$\frac{dP_0}{dr} = -\frac{1}{1+r}\left[\sum_{t=1}^{\infty} \frac{tC_t}{(1+r)^t}\right]$$
The term in square brackets is the duration.

Calculation of duration for a bond with T periods till maturity:

$$D = \frac{1}{P} \sum_{t=1}^{T} tPV(C_t)$$

where P is the current bond price and $PV(\cdot)$ is the present value operator

Modified duration

$$D^* = \frac{D}{1+y}$$

Hence, modified duration is minus the first derivative of the bond price with respect to the interest rate.

Rules for duration:

1. The duration of a zero-coupon bond equals its time to maturity.
2. Holding maturity constant, a bond's duration is higher when the coupon rates is lower.
3. Holding the coupon rate constant, a bonds duration generally increases with its time to maturity. Duration always increases with maturity for bonds selling at par or at a premium to par.
4. Holding other factors constant, the duration of a coupon bond is higher when the bond's yield to maturity is lower.

5. The duration of a level perpetuity is
$$\frac{(1+y)}{y}$$
where y is the yield.

6. The duration of a level annuity is equal to
$$\frac{(1+y)}{y} - \frac{T}{(1+y)^T - 1}$$
where T is the number of payments and y is the annuity's yield per payment period.

7. The duration of a coupon bond equals
$$\frac{1+y}{y} - \frac{(1+y) + T(c-y)}{c\left[(1+y)^T - 1\right] + y}$$

8. The duration of a coupon bond selling at par value is
$$\frac{1+y}{y}\left[1 - \frac{1}{(1+y)^T}\right]$$

Exercise 5.

The term structure is flat with annual compounding. Consider the pricing of a perpetual bond. Let C be the per period cash flow
$$B_0 = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

1. Determine the first derivative of the price with respect to the interest rate.
2. Find the duration of the bond.

Solution to Exercise 5.

1.
$$\frac{dB_0}{dr} = \frac{-C}{r^2}$$

2. Now, we know that with discrete compounding and a flat term structure the duration D has the following relation with the first derivative.
$$\frac{dB_0}{dr} = -\frac{1}{1+r}B_0 D$$
Rewrite the first derivative above as
$$\frac{dB_0}{dr} = -\left(\frac{1}{1+r}\right)\left[\frac{C(1+r)}{r^2}\right]$$
Divide and multiply by $B_0 = \frac{C}{r}$.
$$\frac{dB_0}{dr} = -\left(\frac{1}{1+r}\right)B_0\left[\frac{\frac{-C(1+r)}{r^2}}{B_0}\right]$$
Simplifying the term in square brackets
$$\frac{\left[\frac{C(1+r)}{r^2}\right]}{\frac{C}{r}} = \frac{1+r}{r}$$
This is the duration for a perpetual bond. Observe that it is independent of the coupon payment, but note also this assumes the bond is correctly priced.

6 Convexity

Recall duration – first derivative.

$$\frac{\Delta P}{P} \approx -D^* \Delta y$$

Only an approximation, to be more accurate also account for second order effects.
Convexity: curvature of the relationship between bond prices and interest rates.
Modify above as

$$\frac{\Delta P}{P} = -D^* \Delta y + \frac{1}{2} \times \text{Convexity} \times (\Delta y)^2$$

Calculate convexity of a bond with T periods left as:

$$\text{Convexity} = \frac{1}{P(1+r)^2} \sum_{t=1}^{T} (t + t^2) PV(C_t)$$

where C_t is the cash flow at time t and $PV(\cdot)$ is the present value operator.

Exercise 6.

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Find the bond's current price.
2. Suppose the interest rate changes to 10%, determine the new price of the bond by direct calculation.
3. Instead of direct calculation, use duration to estimate the new price and compare it to the correct price.
4. Use convexity to improve on your estimation using duration.

Solution to Exercise 6.

1. The bond price:

t	$=$	0	1	2	3
C_t	$=$	0	10	10	110

$$NPV = 0 + \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531$$

2. If the interest rate increases to 10%, the bond will be selling at par, equal to 100, which can be confirmed with direct computation:

$$NPV = 0 + \frac{10}{(1+0.1)^1} + \frac{10}{(1+0.1)^2} + \frac{110}{(1+0.1)^3} = 100$$

3. Calculate the bond's duration:

t	C_t	$PV(C_t)$	$tPV(C_t)$
1	10	9.2	9.2
2	10	8.4	16.8
3	110	84.9	254.8
Sum		102.5	280.8
Bondprice			102.531
Duration			2.74

Modified duration:

$$D^* = \frac{D}{1+r} = \frac{2.74}{1.09} = 2.51$$

Let us now calculate the change in the bond price

$$\frac{\Delta P}{P} = -D^* \Delta y = -2.51 \cdot 0.01 = -0.0251$$

Which means theat the bond price changes to:

$$P + \delta P = 102.531 + \left(\frac{\Delta P}{P}\right) P = 102.531 - 0.0251 \cdot 102.531 = 99.957$$

4. Calculate the bond's convexity

t	C_t	$PV(C_t)$	$PV(C_t)(t^2+t)$
1	10	9.2	18.3
2	10	8.4	50.5
3	110	84.9	1019.3
Sum		102.5	1088.1
Bondprice			102.531
Convexity			8.93

Recalculating the change in the bond price using convexity:

$$\frac{\Delta P}{P} = -D^* \Delta y + \frac{1}{2}(\text{Convexity}) = -2.51 \cdot 0.01 + \frac{1}{2} 8.93 (0.01)^2 = -0.0251 + 0.00044 = -0.02465$$

Use this to re-estimate the bond price:

$$P + \delta P = 102.531 \left(1 + \left(\frac{\Delta P}{P}\right) P\right) = 102.531(1 - 0.02465) = 100.0036$$

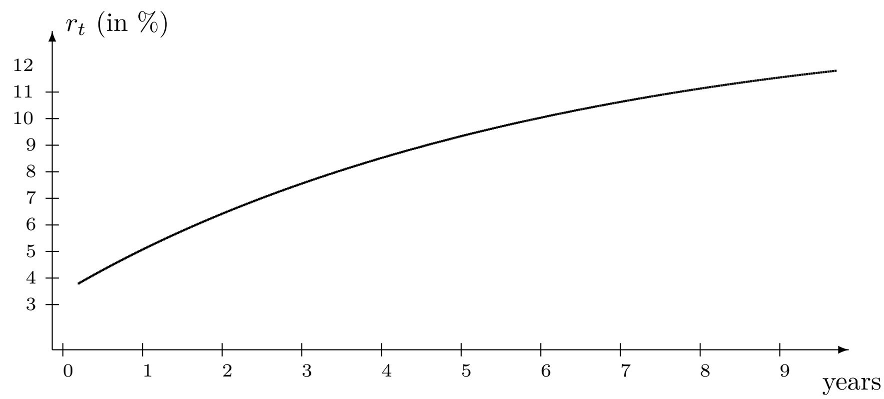
7 Bond Pricing - term structure of interest rates

The simple bond pricing formula

$$B_0 = E[C]\left(\frac{1}{r} - \frac{1}{r(1+r)^T}\right) + \frac{E[F_T]}{(1+r)^T}$$

is only valid when there is a single interest rate (cost of capital) r.

In practice the world is not that simple, one have to deal with the fact that the interest rates used for disounting variew with the time the payment occurs, the *term structure of interest rates*



One will then have to use the term structure for pricing the bond, either in terms of the spot rates r_t or the discound factors d_t.

In terms of spot rates (r_t):

$$B_0 = \sum_{t=1}^{T} \frac{E[C_t]}{(1+r_t)^t} + \frac{E[F_T]}{(1+r_t)^T}$$

In terms of discount factors (d_t):

$$B_0 = \sum_{t=1}^{T} d_t E[C_t] + d_T E[F_T]$$

Exercise 7.

A bond promises the following sequence of payments:

t	=	1	2	3	4
Cashflow X_t	=	10	10	10	110

The interest rates r_t and prices d_t of future risk free cash flows are as follows

t	=	1	2	3	4
r_t	=	5.3%	5.4%	5.6%	5.7%
d_t	=	0.95	0.9	0.85	0.80

Interest rates are compounded annually.

1. Calculate the bond's price

Solution to Exercise 7.

1.

$$\text{Bond Price} = \sum_{t=1}^{4} P_t X_t = 0.95 \cdot 10 + 0.9 \cdot 10 + 0.85 \cdot 10 + 0.8 \cdot 110 = 115$$

7.1 Bond duration under a nonflat term structure with discrete commpounding

Remember the duration as the sensitivity of the bond price to changes in *the* interest rate in the flat term structure setting, where we calculate duration as:

$$D = \frac{1}{B_0} \sum_{t=1}^{T} t PV(C_t)$$

which in the flat term structure setting is:

$$D = \frac{1}{B_0} \sum_{t=1}^{T} \frac{tC_t}{(1+r)^t}$$

With a nonflat term structure we have to adjust this.

First, what are we using duration for? Still want to measure sensitivity to changes in interest rates, but can no longer look at change in *the* interest rate. Therefore look at sensitivity to *parallell changes* in the level of interest rates.

Calculations:

Time t bond price (B_0)

$$B_0 = \sum_i \frac{C_{t_i}}{(1 + r(t, t_i))^{t_i}}$$

Duration (D)

$$D = \frac{1}{B_0} \sum_i t_i PV(C_{t_i})$$

$$D = \frac{1}{B_0} \sum_i t_i d(t, t_i) C_{t_i}$$

$$D = \frac{1}{B_0} \sum_i t_i \frac{C_{t_i}}{(1+r(t,t_i))^{t_i}}$$

An alternative formulation of duration is called the Macaulay duration, and it involves the yield to maturity y of the bond.

$$D = \frac{1}{B_0} \sum_i t_i \frac{C_{t_i}}{(1+y)^{t_i}}$$

Yield to maturity y solves

$$B_0 = \sum_i \frac{C_{t_i}}{(1+y)^{t_i}},$$

hence the above can also be written as

$$D = \frac{\sum_i t_i \frac{C_{t_i}}{(1+y)^{t_i}}}{\sum_i \frac{C_{t_i}}{(1+y)^{t_i}}}$$

When we used duration to measure interest rate sensitivity we also had a second order term involving the bonds *convexity*.

Convexity (Cx) is calculated as:

Using the whole term structure

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) PV(C_{t_i})$$

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) d(t,t_i) C_{t_i}$$

With the Macaulay type of calculation using the yield to maturity y in calculating the present value.

$$Cx = \frac{1}{B_0} \frac{1}{(1+y)^2} \sum_i (t_i + t_i^2) \frac{C_{t_i}}{(1+y)^{t_i}}$$

Exercise 8.

A 10%, two year bond is traded at a price of 90. The current one year spot rate is $r(0,1) = 12\%$ (with discrete, annual compounding). The bond has a face value of 100.

1. Determine the duration and convexity of the bond, using both the full term structure and the Macaulay style calculations.

Solution to Exercise 8.

First need to find the two year spot rate

$$90 = 10d_1 + 110d_2$$
$$d_1 = \frac{1}{1+0.12} = 0.89286$$
$$d_2 = \frac{90 - 10d_1}{110} = 0.73701$$
$$r_2 = 0.16483$$

Here are some of the calculation in a matrix tool

```
> C=[10 110]
C =
   10  110
> r(1)=0.12
r =   0.12000
> d(1)=1/(1+r(1))
```

```
d =   0.89286
> d(2)=(90-10*d(1))/110
d =
  0.89286  0.73701
> d=d'
d =
  0.89286
  0.73701
> BondPrice=C*d
BondPrice = 90
> r(2)=d(2)^(-1/2)-1
r =
  0.12000  0.16483
> y = irr([-BondPrice C],0)
y = 0.16249x
> checkprice=C(1)/(1+y)+C(2)/(1+y)^2
checkprice = 90
```

We calculate duration using the two definitions

```
> Duration=1/BondPrice * ( 1*d(1)*C(1) + 2*d(2)*C(2))
Duration = 1.9008
> Duration=1/BondPrice * ( 1*C(1)/(1+y) + 2*C(2)/((1+y)^2) )
Duration = 1.9044
```

Using the term structure we find duration as

$$D = 1.9008$$

using the Macaulay definition we find

$$D = 1.9044$$

Thus, not a major difference.

We also calculate the convexity for the two definitions

```
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*d(1)*C(1) + (2+2^2)*d(2)*C(2))
Cx = 4.1463
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*C(1)/(1+y) + (2+2^2)*C(2)/(1+y)^2)
Cx = 4.1570
```

Again, not a major difference with the two methods of calculating

8 Summary – bond pricing

Bond: Fixed income security with prearranged payments (coupon C, principal F)

Bonds classified by issuer

- Governments
  - Treasury securities (T-bills, T-bonds)
  - Municipalities
- Corporations
  - Corporate bonds

Bonds classified by coupon terms:

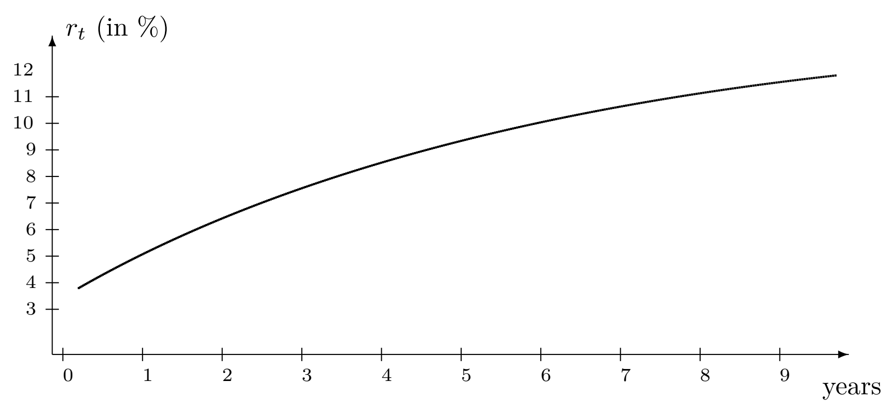
- Zero coupon (only repay principal, issued at discount (below face value))
- Fixed coupon
- Variable coupon – Floating rate bond

Bond price B for a fixed interest rate bond with coupon C and principal F. Required interest rate r:

$$B_0 = C\left(\frac{1}{r} - \frac{1}{r(1+r)^T}\right) + \frac{F}{(1+r)^T}$$

Yield to maturity: Internal rate of return of investment of buying a bond and keeping it to maturity.

Empirically – Interests rate varies with maturity – the term structure of interest rates.



To specify link between interest rate and time to maturity:

- Discount factor d_t – current price of zero coupon bond paying one at time t
- Spot rate r_t – current interest rate for payments recieved at time t.

Bond price when accounting for the term structure:

$$B_0 = \sum_{t=1}^{T} d_t C_t + d_T F_T \text{ (with discount factors)}$$

$$B_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t} + \frac{F_T}{(1+r_T)^T} \text{ (with spot rates)}$$

Duration: Average maturity of a bond.
Calculation of duration for a bond with T periods till maturity:

$$D = \frac{1}{P} \sum_{t=1}^{T} tPV(C_t)$$

where P is the current bond price and $PV(\cdot)$ is the present value operator

References

Jonathan Berk and Peter DeMarzo. *Corporate Finance*. Pearson, fifth edition, 2020.