Bond Portfolios

Bernt Arne Ødegaard

18 April 2023

1 Issues

Source of uncertainty in bond portfolios: Interest rate risk.

Main tool for measuring this: Duration. Convexity.

Passive bond management

- Immunization
- Cash flow matching

Active bond management

- Sources of profits:
  - Indentification of relative mispricing.
  - Interest rate forecasting

2 Duration and convexity

2.1 Duration for a bond

Calculation of duration for a bond with T periods till maturity:

$$D = \frac{1}{P} \sum_{t=1}^{T} t PV(C_t)$$

where P is the current bond price and $PV(\cdot)$ is the present value operator

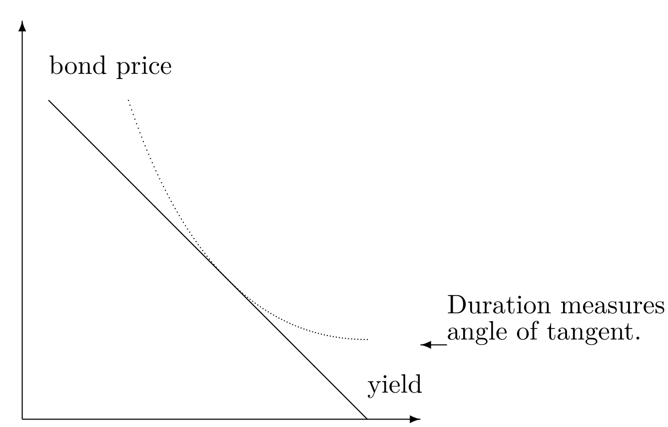
Modified duration

$$D^* = \frac{D}{1+y}$$

Rules for duration:

1. The duration of a zero-coupon bond equals its time to maturity.
2. Holding maturity constant, a bond's duration is higher when the coupon rates is lower.
3. Holding the coupon rate constant, a bonds duration generally increases with its time to maturity. Duration always increases with maturity for bonds selling at par or at a premium to par.
4. Holding other factors constant, the duration of a coupon bond is higher when the bond's yield to maturity is lower.
5. The duration of a level perpetuity is
$$\frac{(1+y)}{y}$$
where y is the yield.
6. The duration of a level annuity is equal to
$$\frac{(1+y)}{y} - \frac{T}{(1+y)^T - 1}$$
where T is the number of payments and y is the annuity's yield per payment period.
7. The duration of a coupon bond equals
$$\frac{1+y}{y} - \frac{(1+y) + T(c-y)}{c\left[(1+y)^T - 1\right] + y}$$
8. The duration of a level annuity is equal to
$$\frac{(1+y)}{y} - \frac{T}{(1+y)^T - 1}$$
where T is the number of payments and y is the annuity's yield per payment period.
9. The duration of a coupon bond selling at par value is
$$\frac{1+y}{y}\left[1 - \frac{1}{(1+y)^T}\right]$$

Duration is related to the change in bond price as a function of the interest rate



If we want to approximate the bond price change from a change in yield, take a Taylor expansion

$$dP = \frac{dP}{dy}dy + \frac{1}{2}\frac{d^2P}{dy^2}(dy)^2 + \text{error}$$

Duration enters into the first term.

2.2 Convexity

Recall duration – first derivative.

$$\frac{\Delta P}{P} = -D^* \Delta y$$

Only an approximation, to be more accurate also account for second order effects.
Convexity: curvature of the relationship between bond prices and interest rates.
Modify above as

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2} \times \text{Convexity} \times \Delta y$$

Calculate convexity of a bond with T periods left as:

$$\text{Convexity} = \frac{1}{P(1+r)^2}\sum_{t=1}^{T}(t+t^2)PV(C_t)$$

where C_t is the cash flow at time t and $PV(\cdot)$ is the present value operator.

If you want a better measure of the change in the price as a function of yield, must also use the second term. This second term is called the *convexity* of the bond.

2.3 Duration and convexity of a portfolio

Portfolio of n assets defined by weights $\{w_i\}_{i=1}^n$ that satisfies $w_i \geq 0 \forall i$ and $\sum_{i=1}^n w_i = 1$.

Both duration and convexity of portfolio can be found as weighted averages of each individual bond in the portfolio

$$D_p = \sum_{i=1}^{n} w_i D_i$$

$$\text{convexity} = \sum_{i=1}^{n} w_i \text{convexity}_i$$

Exercise 1.

Consider an equally weighted portfolio of two bonds, A and B. Bond A is a zero coupon bond with 1 year to maturity. Bond B is a zero coupon bond with 3 years to maturity. Both bonds have face values of 100. The current interest rate is 5%.

1. Determine the bond prices.
2. Your portfolio is currently worth 2000. Find the number of each bond invested.
3. Determine the duration of the portfolio.
4. Determine the convexity of your position.

Solution to Exercise 1.

1. Bond prices Calculate bond prices:

$$P_A = \frac{100}{1.05} = 95.24$$
$$P_B = \frac{100}{1.05^3} = 86.38$$

2. Let n_A be the number of bond A to buy and n_B the number of bond B. Since the fractions are equal, invested 1000 in each bond.

$$n_A = \frac{1000}{95.24} = 10.50$$
$$n_B = \frac{1000}{86.38} = 11.57$$

Want to buy 10.50 A bonds and 11.57 B bonds.

3. Portfolio defined by weights

$$w_A = \frac{1}{2}$$
$$w_B = \frac{1}{2}$$

Since both bonds are zero coupon, duration equals maturity.

$$D_A = 1$$
$$D_B = 3$$

Duration of portfolio then

$$D = w_A D_A + w_B D_B = \frac{1}{2}1 + \frac{1}{2}3 = 2$$

4. Calculating convexity. Bond A:

t	C_t	$PV(C_t)$	$PV(C_t)(t^2+t)$
1	100	95.2	190.5
Sum		95.2	190.5
Bondprice			95.2381
Convexity			1.81

Bond B:

t	C_t	$PV(C_t)$	$PV(C_t)(t^2+t)$
1	0	0.0	0.0
2	0	0.0	0.0
3	100	86.4	1036.6
Sum		86.4	1036.6
Bondprice			86.3838
Convexity			10.88

Convexity of portfolio:

$$\frac{1}{2}1.81 + \frac{1}{2}10.88 = 6.345$$

The portfolio has slightly lower convexity than the obligation.

3 Passive bond management

Two groups of strategies

- Indexation
- Immunization

Indexation Given an index of bond performance
Construct a bond portfolio that matches the index
Works like equity indexing – return to this in that context
Immunization

- Duration/Convexity matching
- Cash flow matching (dedication)

3.1 Liability matching

Consider a pension fund.

Typically, it has a good handle on the future cash flow obligations they are facing, since this is a function of the number of pensioners and the amount to pay each of these each month.

This is the situation for many institutional investors, they are facing a sequence of predictable liabilities.

The problem is then to create a portfolio that best serves those liabilities, by producing a sequence of cash flows covering the liabilities, at the highest possible earnings.

A number of techniques is used to do this. The first involves matching the maturity of the bond portfolio with the obligations (immunization).

3.1.1 Immunization

Suppose you have a portfolio of bonds which is meant to cover your liabilities.

Consider the price of your portfolio as a function of the yield.

If duration of the portfolio is equal to the duration of the liabilities, the portfolio is insured against parallel shifts in the yield curve.

Running an immunized portfolio assumes

- The present value of assets match the present value of liabilities.
- The duration (or interest-rate sensitivity) of the assets must match the duration of the liabilities.
- The convexity of the assets must be larger than the convexity of the liabilities.

Thus, the idea is: One *immunizes* a future payment obligation by creating a bond position with the same duration.

Exercise 2.

A company is facing a cash outflow of 1000 two years from now, which it seeks to immunize. It has identified two bonds, bond A and bond B, which is to be used in this immunization. Bond A is a zero coupon bond with one year remaining to maturity, while bond B is a three year coupon bond with 4% annual coupons. Each bond carries a face value of 100.

The following spot rates apply in the bond bond market: $r_1 = 2\%$, $r_2 = 3\%$, $r_3 = 4\%$ and $r_4 = 4.5\%$, where r_t is the spot interest rate for borrowing over t years, with discrete, annual compounding.

1. Using bonds A and B, find the bond portfolio that best immunizes the company's future payment obligations, based on duration.
2. How does the convexity of the bond portfolio compare to the convexity of the payment obligation? Is the payment obligation perfectly immunized?
3. Suppose that all spot rates fall by one percentage point (i.e. the spot rates change to $r_1 = 1\%$, $r_2 = 2\%$ etc.) Calculate the resultant bond price to check how well the company is protected against a decrease in interest rates.

Solution to Exercise 2.

1. First calculate bond prices

$$P_A = \frac{100}{1.02} = 98.0392$$

$$P_B = \frac{4}{1.02} + \frac{4}{1.03^2} + \frac{104}{1.04^3} = 100.1476$$

Calculate duration:

$$D_A = 1$$

which we don't have to calculate since this is a zero coupon bond with duration equal to the bond maturity.

t	C_t	$PV(C_t)$	$tPV(C_t)$
1	4	3.8	3.8
2	4	3.7	7.4
3	104	92.6	277.8
Sum		100.1	289.0
Bondprice			100.148
Duration			2.8862

$$D_B = 2.8862$$

The duration of the payment obligation is 2 years.
Want to find the portfolio of A and B which has duration equal to 2.

$$w_A D_A + w_B D_B = 2$$

Since $w_A + w_B = 1$, replace w_B with $1 - w_A$:

$$w_A D_A + (1 - w_A) D_B = 2$$

and solve for w_A:

$$w_A = 0.4698$$

Therefore, want to invest 46.98% of the portfolio in bond A and (100-46.98%=53.02%) of the portfolio in bond B. The total amount placed in this bond portfolio is given by the present value of the payment obligation, which is $P_0 = \frac{1000}{1.03^2} = 942.5959$. Let n_A and n_B be the number of bonds (A and B) needed to create a bond portfolio with value 942.5959.

$$n_A = \frac{0.4698 \times 942.5959}{98.04} = 4.5169$$

$$n_B = \frac{(1 - 0.4698) \times 942.5959}{100.1476} = 4.9903$$

2. The convexity for the payment obligation is given by

t	C_t	$PV(C_t)$	$PV(C_t)(t^2 + t)$
1	0	0.0	0.0
2	1000	942.6	5655.6
Sum		942.6	5655.6
Bondprice			942.596
Convexity			5.66

The convexities for the bonds are calculated as

- Bond A:

t	C_t	$PV(C_t)$	$PV(C_t)(t^2 + t)$
1	100	98.0	196.1
Sum		98.0	196.1
Bondprice			98.0392
Convexity			1.92

- Bond B

  Note that the yield to maturity y is 3.9469% for the bond.

t	C_t	$PV(C_t)$	$PV(C_t)(t^2+t)$
1	4	3.8	7.7
2	4	3.7	22.2
3	104	92.6	1111.2
Sum		100.1	1141.1
$Bondprice$			100.148
$Convexity$			10.55

The convexity of the bond portfolio is

$$0.4698 \times 1.92 + 0.5302 \times 10.55 = 6.49$$

Since the convexity of the portfolio is above the convexity of the obligation, conclude that the position is over-immunized in the sense that the value of the bond portfolio will change less for a given change in interest rates compared to the original cash position.

3. For a decrease in interest rates, the new value of the obligation is

$$\frac{1000}{1.02^2} = 961.17$$

The new value of the bond portfolio is

$$P_A n_A + P_B n_B$$
$$P_A = \frac{100}{1.01} =?$$
$$P_b = \frac{4}{1.01} + \frac{4}{1.02^2} + \frac{104}{1.03^3} =?$$
$$Value = \frac{100}{1.01} n_A + \left[\frac{4}{1.01} + \frac{4}{1.02^2} + \frac{104}{1.03^3}\right] n_B = 961.1178$$

The value of the bond portfolio increases (slightly) less in value that the value of the payment obligation.

Exercise 3.

You are working as a bond portfolio manager and is facing the following sequence of liabilities:

Year	1	2	3
Liability	10	200	400

The current term structure of interest rates is observed (with continuous compounding)

time t	1	2	3
spot rate $r(0,t)$	3%	4%	5%

Two bonds are traded: Bond A is a 3 year, 10% coupon bond. Bond B is a 2 year, 5% coupon bond. Both bonds have face values of 100.

1. Find the current bond prices.
2. Find a portfolio of these two bonds that immunizes the liability.
3. What is the convexity of the liability and the immunizing portfolio?

Solution to Exercise 3.

1. Bond Prices

```
> r=[0.03 0.04 0.05]
r =
  0.030000  0.040000  0.050000
> t=[1 2 3]
t =
  1  2  3
> d=exp(-r.*t)
d =
  0.97045  0.92312  0.86071
```

Now price

```
> CflowA=[10 10 110]
CflowA =
   10   10  110
> bA=CflowA*d'
bA = 113.61
> CflowB=[5 105 0]
CflowB =
    5  105    0
> bB= CflowB*d'
bB = 101.78
```

bond Prices

$$B_A = 113.61$$

$$B_B = 101.78$$

2. Duration Bond A

```
> DA =(d(1)*CflowA(1) + 2*d(2)*CflowA(2) + 3*d(3)*CflowA(3))/bA
DA = 2.7480
```

t	C	d(t) C	t d(t)C
1	10	4.8522	4.8522
2	10	9.2312	18.462
3	110	94.678	284.03
Sum			312.20
B_A			113.61
D_A			2.7480

Bond B

```
> DB=(d(1)*CflowB(1) + 2*d(2)*CflowB(2))/(CflowB*d')
DB = 1.9523
```

t	C	d(t) C	t d(t)C
1	5	4.8522	4.8522
2	105	96.927	193.85
Sum			198.71
D_B			1.9523

Obligation has duration equal to

```
> Obl = [10 200 400]
Obl =
   10  200  400
> Dobl =(d(1)*Obl(1) + 2*d(2)*Obl(2) + 3*d(3)*Obl(3))/(Obl*d')
Dobl = 2.6212
```

Want to choose a portfolio of the two assets with the same duration as the obligation

$$w_A D_A + w_B D_B = 2.6212$$

Since

$$w_A + w_B = 1$$

$$w_A D_A + (1 - w_A) D_B = 2.6212$$

$$w_A = \frac{2.6212 - D_B}{D_A - D_B} = \frac{2.6212 - 1.9523}{2.7480 - 1.9523} = 0.8406$$

$$w_B = 1 - w_A = 1 - 0.8406 = 0.1594$$

Choose the number of each of the two assets from:

Present value (obligation):

```
> pvObl = Obl*d'
pvObl = 538.61
```

Split the purchase of this present value on the two bonds according to weights and bond prices:

```
> nA=(pvObl*wA)/bA
nA = 3.9851
> nB=(pvObl*wB)/bB
nB = 0.84353
```

3. Calculate convexity of the obligation and the two bonds using either of the formulas

$$\frac{\sum_i C(t_i)(t_i - t)^2 d(t, t_i)}{P_0}$$

$$\frac{\sum_i C(t_i)(t_i - t)^2 e^{-r(t,t_i)(t_i - t)}}{P_0}$$

$$\frac{\sum_i C(t_i)(t_i - t)^2 e^{-y(t_i - t)}}{P_0}$$

Obligation:

```
> ConvObl=(d(1)*Obl(1) + 2^2*d(2)*Obl(2)+3^2*d(3)*Obl(3))/(Obl*d')
ConvObl = 7.1420
```

Individual bonds:

```
> ConvA = (d(1)*CflowA(1) + 2^2*d(2)*CflowA(2) + 3^2*d(3)*CflowA(3))/bA
ConvA = 7.9104
> ConvB = (d(1)*CflowB(1) + 2^2*d(2)*CflowB(2) )/bB
ConvB = 3.8570
```

and then calculate the convexity of the portfolio by taking the weighted average:

```
> wA=0.8406
wA = 0.84060
> wB=1-wA
wB = 0.15940
> ConvPortf = wA*ConvA + wB*ConvB
ConvPortf = 7.2643
```

3.1.2 Dedicated portfolio

Suppose we are given the sequence of future liabilities as a set of needed cashflows.

One way to cover a set of future liabilities is to invest in a set of bonds that always at least produce the necessary cashflows in the future.

In theory this solves all problems, and provide for the future liabilities (cash flows).

In practice, however, run into a number of problems.

Key problems:

- Available bond maturities may not match the liabilities exactly.
- May need to buy bonds in the future

As soon as there are such mismatches, face *reinvestment risk*.

But let us illustrate the idea in a simple case

Bond selection using linear programming Generally, suppose we want to match liabilities $\{L_t\}_{t=1}^T$.

Given a set of I bonds with current prices B_i, cash flows $X_i(t)$. We want to choose the number of each bond n_i, to minimize cost, at the same time as matching the future liabilities.

The matching problem solves the following linear program

$$\min_{\{n_i\}} \sum_{i=1}^{I} n_i B_i$$

subject to

$$\sum_i n_i X_i(t) \geq L_t \quad \text{for all dates } t$$

Exercise 4.

A pension fund is facing the following set of future liabilities:

	Year		
	1	2	3
Liability	100	100	100

To cover this set of liabilities, the following bonds are available:

Bond no		Current Price	Cash flow in Year 1	2	3
1	Bond 1	100.00	10	110	0
2	Bond 2	95.00	8	8	108
3	Bond 3	105.00	12	12	112
4	Strip 1	94.3396	100		
5	Strip 2	85.7339		100	
6	Strip 3	75.1315			100

1. What are the current (continously compounded) spot rates implied in the strip prices?
2. By investing in the three strips, find how many of each bond one need to match the liabilities.
3. What is the cost of this bond portfolio?
4. Set up the linear program for finding the cost minimizing portfolio that matches the liabilities.
5. You are given the following portfolio that solves this optimal program:

bond no	Bond	n
1	Bond 1	0.8117
2	Bond 2	0
3	Bond 3	0.8929
4	Strip 1	0.8117
5	Strip 2	0
6	Strip 3	0

What is the cost of this portfolio?

6. What does this imply about arbitrage possibilities?
7. What can explain such apparent arbitrage possibilities?

Solution to Exercise 4.

1. Spot rates: Use the zeros to find discount factors

```
> Cflows = [100 0 0; 0 100 0 ; 0 0 100]
Cflows =
  100    0    0
    0  100    0
    0    0  100
> prices = [  94.340  85.734  75.132 ]'
prices =
  94.340
  85.734
  75.132
> d=inv(Cflows)*prices
d =
  0.94340
  0.85734
  0.75132
```

   And then calculate spot rates from the discount factors

```
> r(1)=-log(d(1))
> r(2)=-log(d(2))/2
> r(3)=-log(d(3))/3
r =
  0.058269
  0.076961
  0.095310
```

2. The way to achieve a matching portfolio from the strips is to buy one each of the strips.
3. Cost:

```
> StripPrices =[94.3396 85.7339 75.1315]'
StripPrices =
  94.340
  85.734
  75.132
> n=[1 1 1]
n =
  1  1  1
> cost = n * StripPrices
cost = 255.21
```

4. The program to minimize costs:

$$\min n_1 100 + n_2 95 + n_3 105 + n_4 94.3396 + n_5 85.7339 + n_6 75.1215$$

s.t

$$n_1 10 + n_2 8 + n_3 12 + n_4 100 + n_5 0 + n_6 0 \geq 100$$
$$n_1 110 + n_2 8 + n_3 12 + n_4 0 + n_5 100 + n_6 0 \geq 100$$
$$n_1 0 + n_2 108 + n_3 112 + n_4 0 + n_5 0 + n_6 100 \geq 100$$

5. Cost:

```
> Cflows = [10 110 0; 8 8 108; 12 12 112; 100 0 0; 0 100 0 ; 0 0 100]
Cflows =
   10  110    0
    8    8  108
   12   12  112
  100    0    0
    0  100    0
    0    0  100
> Bondprices=[100 95 105 94.3396 85.7339 75.1315]
Bondprices =
  100.000   95.000  105.000   94.340   85.734   75.132
> w=[0.8117 0 0.8929 0.8117 0 0 ]
w =
  0.81170  0.00000  0.89290  0.81170  0.00000  0.00000
> cost= w*Bondprices'
cost = 251.50
```

To see that this meets the constraints, calculate the cash flow in each of the three time periods:

```
> w*Cflows
ans =
  100.00  100.00  100.00
```

The proposed portfolio exactly meets the payment obligations.

6. See that the cost of this portfolio is lower than the portfolio constructed from the strips.

An arbitrage can be had by going long this portfolio and short the portfolio with the strips, earning $255.21 - 251.50 = 3.71$ per round trip.

If the prices of the coupon bonds were to be consistent with the strips, they should have been:

```
> StripBondPrices =[94.3396 85.7339 75.1315]'
StripBondPrices =
  94.340
  85.734
  75.132
> StripBondCashflows = Cflows = [100 0 0; 0 100 0 ; 0 0 100]
StripBondCashflows =
  100    0    0
    0  100    0
    0    0  100
> d = inv(StripBondCashflows) * StripBondPrices
d =
  0.94340
  0.85734
  0.75132
> CouponBondCashflows=[10 110 0; 8 8 108; 12 12 112]
CouponBondCashflows =
   10  110    0
```

```
    8    8  108
   12   12  112
> ConsistentCouponBondPrices = CouponBondCashflows * d
ConsistentCouponBondPrices =
  103.741
   95.548
  105.756
```

The consistent coupon bond price for the first bond is thus quite off compared to the actual prices reported:

```
ActualCouponBondPrices =
  100
   95
  105
```

The arbitrage opportunity occurs because the coupon bonds are relatively underpriced relative to their predicted price (or the strips being overpriced).

7. The pricing can still be correct, if the prices reflect such market imperfections as transactions costs, liquidity differences and the like. But you need very large imperfections to get such a large price differences.

4 Active bond management

The analyst believes can create value by insights/information that leads to deviations from market valuations.

Discussion of efficient markets relevant here.

Two rought groups of strategies

4.1 Indentification of relative mispricing.

Revisit the previous (linear programming) example.

In there identified a mispricing of some of the bonds. In that setting could construct long-short "arbitrage" strategies.

4.2 Interest rate forecasting

Given a forecast of where interest rates are going, position your portfolio relative to the term structure to profit from this direction.

5 Summary – Bond Portfolios

Bond portfolios – risk – interest rates (level/slope) changes.

Passive bond management

- Immunization
- Cash flow matching

Active bond management

- Indentification of relative mispricing.
- Interest rate forecasting