Financial Econometrics

Problem Set

Exercise 1. *Linear Equations.* [3]

Consider the system of linear equations

$$x_1 + 3x_2 = 0$$
$$x_1 + x_2 = 1$$
$$2x_1 + 4x_2 = 1$$

1. Write this system in matrix form.
2. Determine if the system is well-defined.
3. If it has a solution, find the solution.

Exercise 2. [4]

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

1. Compute $\mathbf{ABC}$, $\mathbf{CAB}$, $\mathbf{BCA}$, $\mathbf{CB'A'}$ and $\mathbf{C'B'A'}$.
2. Verify that $(\mathbf{ABC})' = \mathbf{C'B'A'}$.
3. Find the inverses of these matrices. Verify that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}(\mathbf{AB})^{-1}$
4. Verify that $\text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{CAB})$.
5. Show that $\mathbf{A}(\mathbf{A'A})^{-\mathbf{1}}\mathbf{A'}$ and $\mathbf{B}(\mathbf{B'B})^{-\mathbf{1}}\mathbf{B'}$ are both idempotent. What are the ranks of these two matrices?

Exercise 3.

Consider the linear equation

$$3x_1 + 4x_2 = 5$$
$$4x_1 + 6x_2 = 8$$

Solve this system of equations using a matrix tool.

Exercise 4.

Calculate the sum $1 + 2 + 3 + \cdots + 99 + 100$.

Exercise 5.

Define the following variables:

Real numbers $r = 1$, $s = 10$, $u = 2$.

Vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate

$$r\mathbf{a} + s\mathbf{b}$$

$$s\mathbf{A} + u\mathbf{B}$$

$$B'$$

Calculate the determiants of the three matrices
Which of the three matrices are invertible?
Calculate

$$B^{-1}$$

Solve the linear system

$$\mathbf{Bx} = \mathbf{a}$$

Financial Econometrics

Solutions

Exercise 1. *Linear Equations.* [3]

1. Write the system in matrix form.

$$AX = b$$

where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. To determine if the system is well–defined, we check whether the rank of the matrix $[A]$ equals the rank of the augmented matrix $[A|b]$. The simplest is to use a matrix to do this check:

```
>> b = [0;1;1]
 b = 0 1 1
>> A = [1 3;1 1;2 4]
 A = 1 3
     1 1
      2 4
>> rank(A)
 ans = 2
>> rank([A b])
 ans = 2
```

The rank of $[A]$ equals the rank of $[A|b]$, the system is thus well-defined.

3. To solve the system, again use a matrix tool

```
>> A\b ans = 1.50000 -0.50000
```

Thus,

$$x_1 = 1.5$$
$$x_2 = -0.5$$

Exercise 2. [4]

Exercise 3.

Write this in matrix form by defining

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$
$$b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Let us first check that this system is solvable

```
>> A = [3 4;4 6]
A =
  3  4
  4  6
>> b=[5;8]
b =

  5
  8
>> rank(A)
ans = 2
>> rank ([A b])
ans = 2
```

Note how I create the augmented matrix $[A|b]$ by `[A b]`. The rank of the two is the same. Since **A** is square, we can calculate the solution as

```
>> inverse(A)
ans =
   3.0000  -2.0000
  -2.0000   1.5000
>> x = inverse(A) * b
x =
  -1
   2
```

The solution to the system of equations is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(In this case we calculated the solution by finding the inverse. But you should be aware that solving the system of equations by calculation of the inverse is not the numerically most stable way of doing the calculation. `Matlab/Octave` has built in a direct linear matrix solver, which is invoked by the *left division* operator

```
>> x = A\b
x =
  -1
   2
```

This solves the system of equations directly, and it is usually the way to do this operation, unless one needs the inverse for other purposes.)

Exercise 4.

```
>> [1:100]*ones(100,1)
ans =  5050
```

Exercise 5.

Defining the variables

```
>> r=1
r =  1
>> s=10
s =  10
>> u=2
u =  2
>> a=[1;2;3]
a =
   1
   2
   3
>> b=[3;2;1]
b =

   3
   2
   1
>> c=[1;1;1]
c =
```

```
   1
   1
   1
>> A=[1 1 1; 2 2 2 ; 3 3 3 ]
A =
   1   1   1
   2   2   2
   3   3   3
>> B=[3 2 3; 2 4 5; 1 1 1]
B =
   3   2   3
   2   4   5
   1   1   1
>> C= ones(3,3)
C =
   1   1   1
   1   1   1
   1   1   1
>> I3=eye(3)
I3 =
   1   0   0
   0   1   0
   0   0   1
```

Doing the calculations

```
>> r*a+s*b
ans =
   31
   22
   13
>> s*A+u*B
ans =
   16   14   16
   24   28   30
   32   32   32
```

Matrix transpose and inverse

```
>> B'
ans =
   3   2   1
   2   4   1
   3   5   1
>> inv(B)
ans =
   0.33333  -0.33333   0.66667
  -1.00000   0.00000   3.00000
   0.66667   0.33333  -2.66667
```

Calculating determinants

```
>> det(A)
ans = 0
>> det(B)
ans = -3.0000
>> det(C)
ans = 0
```

To see which matrices are invertible we calculate the ranks of the three matrices. It is only when the matrices are square and have ranks equal their dimensions that they are invertible.

```
>> rank(A)
ans =  1
>> rank(B)
ans =  3
>> rank(C)
ans =  1
```

This could alternatively have been seen from the determinants, only when determinants are nonzero can the system be solved.

Solving the linear system

$$\mathbf{Bx} = \mathbf{a}$$

```
>> x=inv(B)*a
x =
   1.6667
   8.0000
  -6.6667
```