Restrictions involving second moments

Approach to estimation that uses restrictions coming from second moments.
Useful diagnostic for many classes of models
Way that one argue about them: starting with the economics, and then derive bounds which have to hold in the data.

- the most cited early work using this approach, the Shiller "excess volatility" papers. In this setting it is very easy to follow the arguments.
- Similar approach to characterize the "pricing operator" m_t, the Hansen and Jagannathan [1991] type of bounds

Both cases: the *method* is the important
Papers are early ones where one can follow the arguments relatively easy.

Intuition: Sharpe Ratio

One reason for why these kinds of bounds are natural to finance researchers:

In finance we are used to think in terms of second moments, one of the first tools of asset pricing is the "Sharpe Ratio",

$$SR_p = \frac{E[r_p]}{\sigma(r_p)}$$

the ratio of expected return to standard deviation of return.

One often talks about "return in units of risk" in this context.

Finance researchers therefore have some intuition about the kind of relationships we can expect from second moment restrictions.

The classical variance bounds tests

Debate started by Shiller [1981] and Leroy and Porter [1981].
Are markets rational in the sense of correctly pricing assets.
We further specify "correctly" as saying that the price should be an estimate of the "fundamental" value of an asset.
Let us look at how this question is asked.

Some economic theory.

Valuation equation

$$V = \sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^{t} d_t$$

If the dividends are risky

$$V = \sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^{t} E[d_t]$$

If interest rates are not constant

$$V = \sum_{t=1}^{\infty} \left(\prod_{k=1}^{t} \left(\frac{1}{1+r_k} \right) \right) E[d_t]$$

Example model with this implication: Lucas [1978] model.

$$V_t U'(C_t) = \beta E_t \left[U'(C_{t+1}) (V_{t+1} + d_{t+1}) \right]$$

One solution to this difference equation is that:

$$V_t = \sum_{j=1}^{\infty} \beta^j E_t \left[\left(\frac{U'(C_{t+j})}{U'(C_t)} \right) d_{t+j} \right]$$

If we have risk neutrality (a linear utility function), $U'(\cdot)$ is a constant, and this reduces to

$$V_t = \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+j} \right]$$

This kind of relationship gives formal statements of the "fundamental" value of an asset.
The question can be transformed into return form, by using the identity

$$1 + E_t[R_{i,t+1}] = \frac{E_t\left[V_{i,t+1} + d_{i,t+1}\right]}{V_{i,t}}$$

Go throught the Grossman and Shiller [1981] paper, following their notation.

$$\max V_t = \sum_k E_t \left[\beta^k U(C_{t+k}) \right]$$

$$\beta = \frac{1}{1+r}$$

FOC for optimum

$$U'(C_t) P_{it} = \beta E \left[U'(C_{t+1})(P_{it+1} + d_{it+1}) | I_t \right]$$

Rewrite as

$$1 = E \left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \frac{(P_{it+1} + d_{it+1})}{P_{it}} | I_t \right]$$

or

$$1 = E[S_t R_{it} | I_t]$$

where

$$S_t = \beta \frac{U'(C_{t+1})}{U'(C_t)} \quad \text{and} \quad R_{it} = \frac{(P_{it+1} + d_{it+1})}{P_{it}}$$

Perfect foresight stock prices: Suppose we know all our future consumption, and that we also know we know we would never sell the stock, only hold stock for dividend. Then

$$P_{it} = \sum_{k=0}^{\infty} \beta^k \frac{U'(C_{t+k+1})}{U'(C_{t+k})} E[d_{it+k}|I_t]$$

If we further knew what dividends will be:

$$P_{it} = \sum_{k=0}^{\infty} \beta^k \frac{U'(C_{t+k+1})}{U'(C_{t+k})} d_{it+k}$$

This is the perfect foresight price.

Given an utility specification

$$U(C_t) = \frac{1}{1-A} C_t^{1-A} \qquad\qquad 0 < A < \infty$$

$$\mu = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-A}$$

For observed historical dividends and consumption, can estimate the "perfect foresight" prices as

$$P_{it} = \sum_{k=0}^{T} \beta^k \left(\frac{C_{t+k+1}}{C_{t+k}} \right)^{-A} d_{it+k} + \beta^T \left(\frac{C_{t+T+1}}{C_{t+T}} \right)^{-A} P_{it+T}$$

These are constructed this way because we have a limited set of observations, need to "cut off" at some horizon T.

Grossman and Shiller [1981] calculate p_t^* from the actual observed dividends. They have actual observed dividends $\{d_t\}$ for the period 1889 to 1979. Given a discount factor β, we can use these to calculate the *ex post* perfect foresight price, where we have to cut off the observations at the last observed price.
Here T is the last observation of 1979.
Most of Shillers early research plotted these time series against actual observed prices, and just by eyeballing them, it seems obvious that actual prices are more volatile than these *ex post rational* prices. See the next figures.

Figure from Shiller 81

FIGURE 1

Note: Real Standard and Poor's Composite Stock Price Index (solid line p) and *ex post* rational price (dotted line p^*), 1871–1979, both detrended by dividing a long-run exponential growth factor. The variable p^* is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter. Data are from Data Set 1, Appendix.

Figure from Grossman Shiller 81

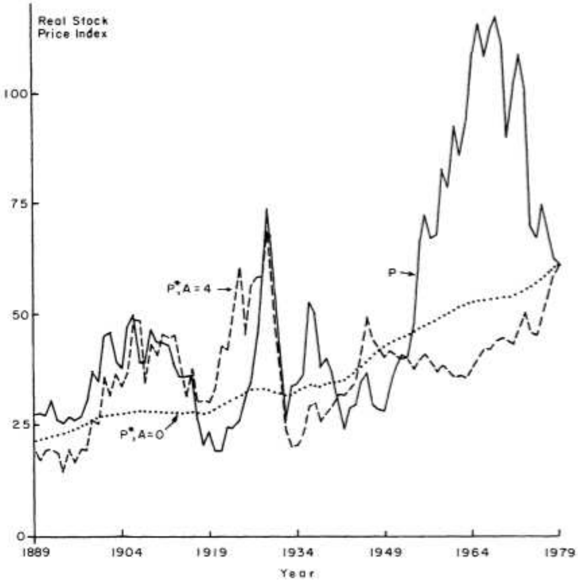

FIGURE 1. ACTUAL AND PERFECT FORESIGHT STOCK PRICES, 1889–1979

Note: The solid line P_t is the real Standard and Poor Composite Stock Price Average. The other lines are: P_t^* (as defined by expression (6) and (7), the present value of

From intuition to formal tests: The variance bound inequalities

The intuition from the pictures can be formally tested by comparing the sample variances of the two series.
Consider this estimate of the prices.

$$p_t = \sum_{j=1}^{\infty} \beta^j E[d_{t+j}]$$

The market price is assumed to be the markets' "best assessment" of the present value of discounted future expted dividends.
Define the "ex post" market fundamental as the equivalent relation, based on observed data.

$$p_t^* = \sum_{j=1}^{\infty} \beta^j d_{t+j}$$

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This should be what the market is estimating.
At time t, the market price is the markets *best estimate* of p_t^*.

$$p_t = E[p_t^* | I_t],$$

where I_t is the markets information set at time t.
Write the market price at time t as the true price (market fundamental) p_t^*, and a forecast error ε_t.

$$p_t^* = p_t + \varepsilon_t$$

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This forecast error is independent of anything in the markets information set at time t.
This independence implies that $\text{cov}(p_t, \varepsilon_t) = 0$, and hence

$$\text{var}(p_t^*) = \text{var}(p_t) + \text{var}(\varepsilon_t)$$

or

$$\text{var}(p_t^*) \geq \text{var}(p_t),$$

The variance of the markets forecast should be lower than the variance of what the market actually is forecasting.
This is the classical "variance bound"

Follow this intuition to get another bound.
This is the basis of the test of West [1988].
Consider the *econometrician*'s information set, call it Z_t.
(S)he will always have *less* information than the market.
Suppose that the econometrician tries to estimate the *fundamental* price p_t^* using The *Econometrician*'s information set.

$$\widehat{p}_t = E[p_t^* | Z_t],$$

$$\widehat{p}_t = E[p_t^* | Z_t],$$

or

$$\widehat{p}_t = p_t^* + \widehat{\varepsilon}_t$$

where again $\widehat{\varepsilon}_t$ is an estimation error.
We can now apply iterated expectations:

We had that

$$p_t = E[p_t^*|I_t],$$

where I_t is the markets information set.
what the econometrician observes is a subset of the market information set.

$$Z_t \in I_t$$

By iterated expectations

$$\begin{aligned} p_t &= E[p_t^*|I_t] \\ &= E[E[p_t^*|Z_t]|I_t] \\ &= E[\widehat{p}_t|I_t] \end{aligned}$$

$$p_t = E[\widehat{p}_t | I_t]$$

We can now use the same technique as before:

$$p_t = \widehat{p}_t + \tilde{\varepsilon}_t$$

$\tilde{\varepsilon}_t$ is again a error with $\text{cov}(\widehat{p}_t, \tilde{\varepsilon}_t)$, and we have

$$\text{var}(p_t) = \text{var}(\widehat{p}_t) + \text{var}(\tilde{\varepsilon}_t)$$

or

$$\text{var}(p_t) \geq \text{var}(\widehat{p}_t)$$

So we get both an upper and lower bound on the variance of the observed price.

$$\text{var}(p_t^*) \geq \text{var}(p_t) \geq \text{var}(\widehat{p}_t)$$

Bounding the Intertemporal marginal rate of substitution.

Recall the pricing equation.

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

In a consumption based asset pricing model we will have $m_t = \frac{u'(c_{t+1})}{u'(c_t)}$. By parameterising $u(c_t)$, this was tested by Hansen and Singleton [1982].
But it is also possible to infer properties of m_t without making further assumptions. We can view this as a non-parametric approach, the properties of m_t identified will have to hold for all candidate m_t parameterisations.

Derivation of the bound with a risk free asset

Consider

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

Suppose we have a risk free asset $R_{f,t}$. Then

$$E_t[m_{t+1}R_{f,t}] = 1$$

Sincd $R_{f,t}$ is constant we can move it outside the expectation.

$$E_t[m_{t+1}]R_{f,t} = 1$$

Subtract the two to get the excess return $r_{i,t+1} = R_{i,t+1} - R_{f,t}$.

$$E_t[m_{t+1}R_{i,t+1}] - E_t[m_{t+1}]R_{f,t} = 1 - 1 = 0$$

$$E_t[m_{t+1}r_{i,t+1}] = 0$$

$$E_t[m_{t+1} r_{i,t+1}] = 0$$

Using

$$0 = E_{t-1}[m_t r_{it}] = \text{cov}_{t-1}(m_t, r_{it}) + E_{t-1}[m_t] E_{t-1}[r_{it}]$$

we have

$$\text{cov}_{t-1}(m_t, r_{it}) = E_{t-1}[m_t] E_{t-1}[r_{it}]$$

Now use the fact that $\text{cov}(x, y) = \sigma(x)\sigma(y)\rho(x, y)$ to get:

$$\rho(m_t, r_{it})\sigma(m_t)\sigma(r_{it}) = E_{t-1}[m_t] E_{t-1}[r_{it}]$$

$$\rho(m_t, r_{it})\sigma(m_t)\sigma(r_{it}) = E_{t-1}[m_t]E_{t-1}[r_{it}]$$

By the definition of correlation, $\rho > -1$. This implies that

$$-1\sigma(m_t)\sigma(r_{it}) \leq E_{t-1}[m_t]E_{t-1}[r_{it}]$$

$$\frac{\sigma(m_t)}{E[m_t]} \geq \frac{E[r_{it}]}{\sigma(r_{it})}$$

Since this will hold for any i, we get that

$$\frac{\sigma(m_t)}{E[m_t]} \geq \max_i \frac{E[r_{it}]}{\sigma(r_{it})}$$

Note that the expression in returns is a Sharpe Ratio.
This is one of the results in the Hansen and Jagannathan [1991] paper.

A similar bound without a risk free asset

Let us now use the more standard asset pricing setup in terms of notation.
The price $\mathbf{q}$ of an asset is a product of the cash flows $\mathbf{x}$ and the stochastic discount factor y. We have that

$$\mathbf{q} = E[y\mathbf{x}]$$

We also know that

$$E[y\mathbf{x}] = E[y]E[\mathbf{x}] + \text{cov}(y, \mathbf{x})$$

Question: Is $y = \mathbf{x}'E[\mathbf{x}\mathbf{x}']^{-1}\mathbf{q}$ a candidate discount factor?

Regress the discount factor y on the observed asset returns $\mathbf{x}$.

$$y = a + \mathbf{x}\mathbf{b} + e$$

By definition,

$$\mathbf{b} = \text{cov}(\mathbf{x}, \mathbf{x})^{-1}\text{cov}(\mathbf{x}, y)$$

This can only be estimated if we know y.
But when y is the true stochastic discount factor, we know that

$$\mathbf{q} = E[y\mathbf{x}] = E[y]E[\mathbf{x}] + \text{cov}(\mathbf{x}, y)$$

or

$$\text{cov}(\mathbf{x}, y) = \mathbf{q} - E[y]E[\mathbf{x}]$$

Substitute for $\text{cov}(\mathbf{x}, y)$:

$$\mathbf{b} = \text{cov}(\mathbf{x}, \mathbf{x})^{-1}\left(\mathbf{q} - E[y]E[\mathbf{x}]\right)$$

Note, that the only unobservable factor here is the mean $E[y]$. All others are observable.
From

$$y = a + \mathbf{x}'\mathbf{b} + e$$

we get

$$\text{var}(y) = \text{var}(\mathbf{x}'\mathbf{b}) + \text{var}(e)$$

Implying that

$$\text{var}(y) \geq \text{var}(\mathbf{x}'\mathbf{b})$$

Hence, substituting for **b**

$$\text{var}(y) \geq \text{var}\left(\mathbf{x}'\left\{\text{cov}(\mathbf{x},\mathbf{x})^{-1}(\mathbf{q} - E[y]E[\mathbf{x}])\right\}\right)$$

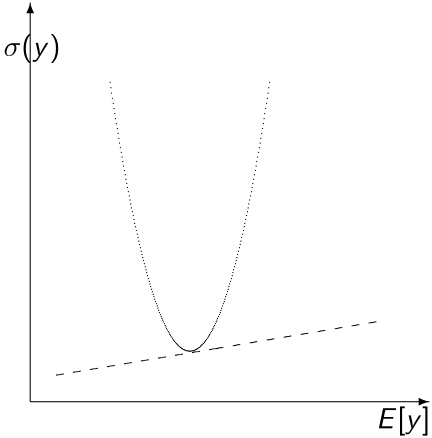
This is in the case where we don't have a risk free asset.

If we have a risk-free asset, the bound is as we showed earlier,

$$\frac{\sigma(y)}{E[y]} \geq \frac{E[r]}{\sigma(r)},$$

where r is the return on some risky asset.

The variance bounds will be shown like the following picture



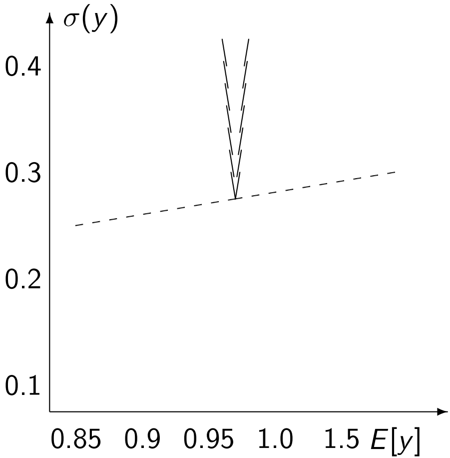
The admissible area for the discount factors will be above the dashed line in the case of a risk free asset. The curve defines the area in the case of no risk free asset.

How do we go about implementing the variance bounds? The bounds are written in terms of expectations and variances. To implement, we need to estimate sample means and variances. The pricing formulas are written in terms of *conditional* pricing functions. We can try to look at

- Conditional Gallant et al. [1990] or
- Unconditional Hansen and Jagannathan [1991] variance bounds.

To use unconditional expectations we need to assume some kind of stationarity on the stochastic process generating returns. If we make these assumptions, we can estimate the expectation and variance from their sample counterparts.

Using sample averages on US data, the results are as shown in the figure below.



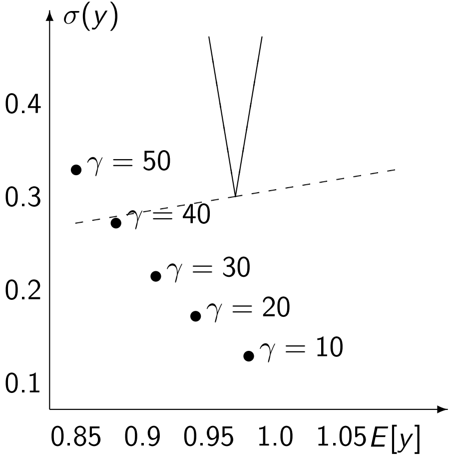
The Equity Premium puzzle of Mehra and Prescott [1985] is a famous result. Essentially, it says that the return on equity is too high to be justified in a representative agent setting. The necessary risk aversion of the utility function of the representative agent is way too high relative to what one estimate for individuals.
The Equity Premium puzzle of Mehra and Prescott [1985] can be illustrated in the imrs bound setting. In their setting the discount factor y is

$$y = \beta \frac{U(c_t + 1)}{U(c_t)}$$

and the utility function is

$$U(c) = \frac{1}{1-\gamma} c^{1-\gamma}$$

To estimate mean and standard deviation of this discount factor, we pick values of β and γ, calculate a time series of $\beta\frac{U'(c_t+1)}{U'(c_t)}$ and estimate its mean and standard deviation.



This is the Equity premium puzzle. To get into the admissible region we need a risk aversion coefficient γ of 40.

The original pricing equality is that

$$\pi(\mathbf{x}) = E[y\mathbf{x}]$$

We can sharpen this by imposing that the pricing operator must be positive:

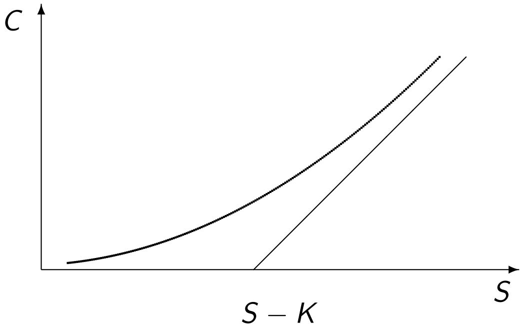
$$\pi(\mathbf{x}) = E[\max(y, 0)\mathbf{x}]$$

Under no arbitrage, the stochastic discount factor must be positive. Imposing this will sharpen the variance bounds.

Good-Deal Asset Pricing Bounds

Bounds involving the "pricing factor" m - not just for investigating the deep questions of macrofinancial asset pricing.
Here: An example application to option pricing, due to Cochrane and Saá-Requejo [1999].

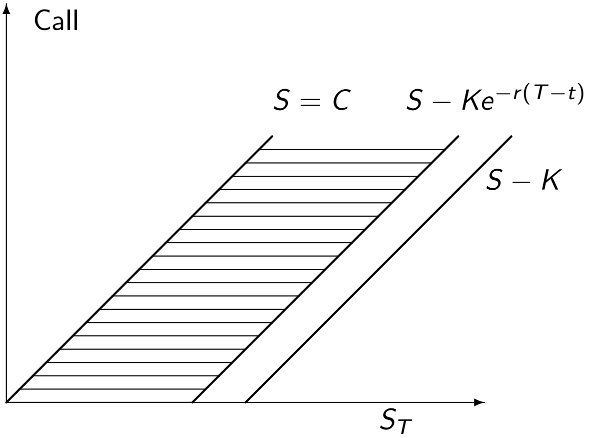
In option pricing, a call option is the present value of the future payoff $(X - K)^+$.
Under the ideal conditions, calculate the exact Black Scholes price:



We know that there also exist a corresponding pricing factor m^* which would also price the call

$$c = E[m^*(S - K)^+]$$

Ideal conditions of the Black Scholes model (continous trading, no transaction costs) are not always there.
"Preference free" arbitrage arguments to put bounds on the option price:



Can we do better?
The suggestion of Cochrane and Saá-Requejo [1999] is to look at candidate "pricing kernels" $m()$.
We know that there is some m that will price the option

$$c = E[m(S-K)^+]$$

Can we establish upper and lower bounds on c by restricing the set of feasible m?

An obvious way of doing that is to use prices $\mathbf{p}$ of other financial contracs (basis assets) with future payoffs $\mathbf{x}$

$$\mathbf{p} = E[m\mathbf{x}]$$

With a set of basis assets one sould search for the m that minimizes the program

$$\underline{C} = \min_{\{m\}} E\left[m(S-K)^{+}\right]$$

subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

A similar program could be used to establish an upper bound

$$\overline{C} = \max_{\{m\}} E\left[m(S-K)^{+}\right]$$

subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

How to make the bounds tighter?
For one thing we can add nonnegative state prices

$$m \geq 0$$

(This follows from the absense of arbitrage in complete markets, where a negative state price would generate infinite profits. In less complete markets it is still hard to think about getting money today for a possible future payoff...)

But, can also argue economically. (This is where the variance bound appears)

Cochrane and Saá-Requejo [1999] introduce the notion of "good deals", trading opportunities with very high Sharpe ratios. Such "good deals" will be traded away

If we can establish an upper bound h on the Sharpe Ratio

$$\frac{\left|R^h\right|}{\sigma(R^h)} = h$$

we can use this as an additional inequality constraint in the optimization programs.

$$h \geq \frac{\sigma(m)}{E[r_m]}$$

If there exists a risk free asset with (gross) returns R_f, will have

$$\text{price of unit payoff} = E[m \cdot 1] = E[m] = \frac{1}{R_f}$$

Hence

$$h \geq \frac{\sigma(m)}{\frac{1}{R_f}}$$

or

$$\frac{h}{R_f} \geq \sigma(m)$$

So we have a "variance bound" on the pricing kernel, which translates into "good deal" bounds

$$\underline{C} = \min_{\{m\}} E\left[m(S-K)^+\right]$$

$$\overline{C} = \max_{\{m\}} E\left[m(S-K)^+\right]$$

both subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

$$m > 0$$

$$\frac{h}{R_f} \geq \sigma(m)$$

Example application (from Cochrane and Saá-Requejo [1999])

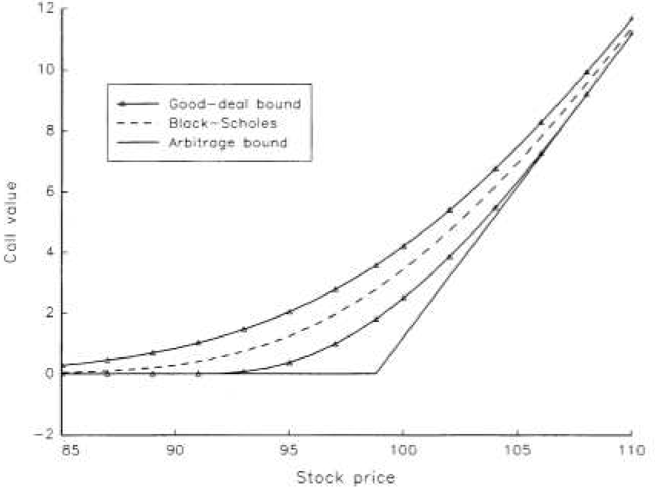


FIG. 1.—Option price bounds as a function of stock price. Options have three months to expiration and strike price $K = \$100$. The bounds assume no trading until expiration and a discount factor volatility bound $h = 1.0$ corresponding to twice the market Sharpe ratio. The stock is lognormally distributed with parameters calibrated to an index option.

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