Restrictions involving second moments

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Introduction

We now consider an approach to estimation that uses restrictions coming from second moments.

This has turned out to be a useful diagnostic for many classes of models.

They are also interesting from a theoretical viewpoint because of the way that one argue about them, starting with the economics, and then derive bounds which have to hold in the data.

We will start by looking at the some of the most cited early work using this approach, the Shiller "excess volatility" papers. In this setting it is very easy to follow the arguments, particularly those that develop the volatility bounds.

We then look at using a similar approach to characterize the "pricing operator" m_t, using the Hansen and Jagannathan (1991) paper.

In both cases it is the *method* which is interesting, the papers that I give references to are not the most recent work in this area, but they are early papers where one can follow the arguments relatively easy.

1 Intuition: Sharpe Ratio

One reason for why these kinds of bounds are natural to finance researchers:

In finance we are used to think in terms of second moments, one of the first tools of asset pricing is the "Sharpe Ratio",

$$SR_p = \frac{E[r_p]}{\sigma(r_p)}$$

the ratio of expected return to standard deviation of return.

One often talks about “return in units of risk” in this context.

Finance researchers therefore have some intuition about the kind of relationships we can expect from second moment restrictions.

2 The classical variance bounds tests

This section covers a debate started by Shiller (1981) and Leroy and Porter (1981). The basic question is whether markets are rational in the sense of correctly pricing assets. We further specify "correctly" as saying that the price should be an estimate of the "fundamental" value of an asset.

Let us look at how this question is asked.

2.1 Some economic theory.

Remember the basic valuation equation from a standard finance course, the value of a stock is the sum of future dividends.

$$V = \sum_{t=1}^{\infty} \left(\frac{1}{1+r}\right)^t d_t$$

If the dividends are risky, we replace the dividends d_t with their expected value.

$$V = \sum_{t=1}^{\infty} \left(\frac{1}{1+r}\right)^t E[d_t]$$

If we don't have constant interest rates, we use another discount factor.

$$V = \sum_{t=1}^{\infty} \left(\prod_{k=1}^{t} \left(\frac{1}{1+r_k}\right)\right) E[d_t]$$

We can derive this kind of asset pricing relationship in a full equilibrium model. Let us look at the case of the Lucas (1978) model. There, we have seen before that the fundamental asset pricing relation is:

$$V_t U'(C_t) = \beta E_t \left[U'(C_{t+1})\left(V_{t+1} + d_{t+1}\right)\right] v$$

One solution to this difference equation is that:

$$V_t = \sum_{j=1}^{\infty} \beta^j E_t \left[\left(\frac{U'(C_{t+j})}{U'(C_t)}\right) d_{t+j}\right]$$

Note that if we have risk neutrality, (a linear utility function), $U'(\cdot)$ is a constant, and this reduces to

$$V_t = \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+j}\right]$$

This kind of relationship gives formal statements of the "fundamental" value of an asset.

The question can be transformed into return form, by using the identity

$$1 + E_t[R_{i,t+1}] = \frac{E_t \left[V_{i,t+1} + d_{i,t+1}\right]}{V_{i,t}}$$

2.2 The Specifics of the Grossman and Shiller (1981) paper

Let us go throught the Grossman and Shiller (1981) paper, following their notation.

$$\max V_t = \sum_k E_t \left[\beta^k U(C_{t+k})\right]$$

$$\beta = \frac{1}{1+r}$$

FOC for optimum

$$U'(C_t)P_{it} = \beta E\left[U'(C_{t+1})(P_{it+1} + d_{it+1})|I_t\right]$$

Rewrite as

$$1 = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \frac{(P_{it+1} + d_{it+1})}{P_{it}} |I_t\right]$$

or

$$1 = E[S_t R_{it}|I_t]$$

where

$$S_t = \beta \frac{U'(C_{t+1})}{U'(C_t)}$$

and

$$R_{it} = \frac{(P_{it+1} + d_{it+1})}{P_{it}}$$

Perfect foresight stock prices: Suppose we know all our future consumption, and that we also know we know we would never sell the stock, only hold stock for dividend. Then

$$P_{it} = \sum_{k=0}^{\infty} \beta^k \frac{U'(C_{t+k+1})}{U'(C_{t+k})} E[d_{it+k}|I_t]$$

If we further knew what dividends will be:

$$P_{it} = \sum_{k=0}^{\infty} \beta^k \frac{U'(C_{t+k+1})}{U'(C_{t+k})} d_{it+k}$$

This is the perfect foresight price.

Given an utility specification

$$U(C_t) = \frac{1}{1-A} C_t^{1-A} \qquad\qquad 0 < A < \infty$$

$$\beta \frac{U'(C_{t+1})}{U'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-A}$$

For observed historical dividends and consumption, can estimate the "perfect foresight" prices as

$$P_{it} = \sum_{k=0}^{T} \beta^k \left(\frac{C_{t+k+1}}{C_{t+k}}\right)^{-A} d_{it+k} + \beta^T \left(\frac{C_{t+T+1}}{C_{t+T}}\right)^{-A} P_{it+T}$$

These are constructed this way because we have a limited set of observations, need to "cut off" at some horizon T.

Grossman and Shiller (1981) calculate p_t^* from the actual observed dividends. They have actual observed dividends $\{d_t\}$ for the period 1889 to 1979. Given a discount factor β, we can use these to calculate the *ex post* perfect foresight price, where we have to cut off the observations at the last observed price.

Here T is the last observation of 1979.

Note that as j increases, $\beta^j \to 0$. As we go far enough into the future, we don't care about those future dividend payments, their impact on todays price will be negligible.

To use the actual prices in this way to generate the terminal price, we are assuming rational expectations and stationary prices.

Most of Shillers early research plotted these time series against actual observed prices, and just by eyeballing them, it seems obvious that actual prices are more volatile than these *ex post rational* prices. See figure 1, where the dashed line is the ex post rational price, and the solid is the observed prices. The observed ones clearly looks more volatile.

Actual stock prices clearly more variable than predicted ones.

The conclusion that Shiller made was that market does not seem to be rational, the prices we observe are much to volatile to be explained by rational investors.

Figure 1 Shiller 81

FIGURE 1

Note: Real Standard and Poor's Composite Stock Price Index (solid line p) and *ex post* rational price (dotted line p^*), 1871–1979, both detrended by dividing a long-run exponential growth factor. The variable p^* is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter. Data are from Data Set 1, Appendix.

(from Shiller (1981))

2.3 The variance bound inequalities

This intuition from the pictures above is also formally tested by comparing the sample variances of the two series.

To simplify matters, consider this estimate of the prices.

$$p_t = \sum_{j=1}^{\infty} \beta^j E[d_{t+j}]$$

The market price is assumed to be the markets' "best assessment" of the present value discounted future dividends.

Define the "ex post" market fundamental as the equivalent relation, based on observed data.

$$p_t^* = \sum_{j=1}^{\infty} \beta^j d_{t+j}$$

Now, this should be what the market is estimating, i.e, at time t, the market price is the markets *best estimate* of p_t^*.

$$p_t = E[p_t^* | I_t],$$

where I_t is the markets information set at time t. We can thus write the market price at time t as the true price (market fundamental) p_t^*, and a forecast error ε_t.

$$p_t^* = p_t + \varepsilon_t$$

This forecast error is independent of anything in the markets information set at time t. This independence implies that $\text{cov}(p_t, \varepsilon_t) = 0$, and hence

$$\text{var}(p_t^*) = \text{var}(p_t) + \text{var}(\varepsilon_t)$$

Figure 2 Grossman Shiller 81

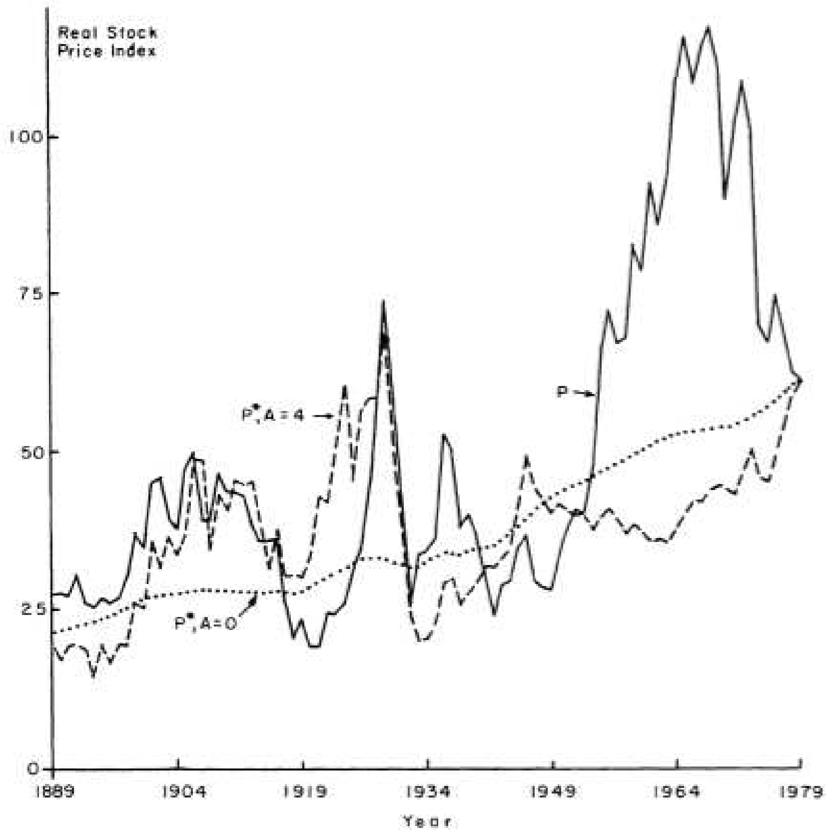


FIGURE 1. ACTUAL AND PERFECT FORESIGHT STOCK PRICES, 1889–1979

Note: The solid line P_t is the real Standard and Poor Composite Stock Price Average. The other lines are: P_t^* (as defined by expression (6) and (7), the present value of actual subsequent real dividends using the actual stock price in 1979 as a terminal value. With $A=0$ (dotted line) the discount rates are constant, while with $A=4$ (dashed line) they vary with consumption.

(from Grossman and Shiller (1981))

or

$$\mathrm{var}(p_t^*) \geq \mathrm{var}(p_t),$$

the variance of the markets forecast should be lower than the variance of what the market actually is forecasting.

Let us follow this intuition to get another bound. This is the basis of the test of West (1988). Consider an econometrician's information set, call it Z_t. (S)he will always have *less* information than the market. Suppose now the econometrician tries to estimate the *fundamental* price p_t^* using *his* information set.

$$\widehat{p}_t = E[p_t^*|Z_t],$$

or

$$\widehat{p}_t = p_t^* + \widehat{\varepsilon}_t$$

where again $\widehat{\varepsilon}_t$ is an estimation error.

We can now apply iterated expectations:

We had that

$$p_t = E[p_t^*|I_t],$$

where I_t is the markets information set. Remember that what the econometrician observes is a subset of the market information set.

$$Z_t \in I_t$$

By iterated expectations

$$\begin{aligned} p_t &= E[p_t^*|I_t] \\ &= E[E[p_t^*|Z_t]|I_t] \\ &= E[\widehat{p}_t|I_t] \end{aligned}$$

We can now use the same technique as before:

$$p_t = \widehat{p}_t + \tilde{\varepsilon}_t$$

$\tilde{\varepsilon}_t$ is again a error with $\mathrm{cov}(\widehat{p}_t, \tilde{\varepsilon}_t)$, and we have

$$\mathrm{var}(p_t) = \mathrm{var}(\widehat{p}_t) + \mathrm{var}(\tilde{\varepsilon}_t)$$

or

$$\mathrm{var}(p_t) \geq \mathrm{var}(\widehat{p}_t)$$

So we get both an upper and lower bound on the variance of the observed price.

$$\mathrm{var}(p_t^*) \geq \mathrm{var}(p_t) \geq \mathrm{var}(\widehat{p}_t)$$

This gives us several interesting testable implications about the variance of prices.

2.4 Problems with the "first-generation" of tests.

Since Shiller conclusion does not really agree with our assumptions about rational investors in the markets, this conclusion was attacked in a number of ways. Let us look at some.

- Econometric problem: Short sample biases in testing the differences in variances. It was shown that there is a bias in testing for differences in favour of rejecting the null to often. However, not really large enough to explain the huge differences.

- The cutoff problem, replacing p_T^* with an estimated value may also induce biases.
- Econometric problem: Is the data series $\{d_t, p_t\}$ stationary?

  This is the *unit root* problem. This is potentially a serious problem. If the two series aren't stationary, the assumptions for doing estimation does not exist, in particular, the variance does not exist. We have to transform the data series in a way that induces stationarity.

  In some of the original paper they used a simple de-trending procedure, but this was heavily criticised.

  A better way of doing this was explored by Campbell and Shiller (1987) by doing transformations that induced stationarity and a VAR specification, and this paper still finds violations of the variance bounds.
- Conceptual problem: The tests for differences in variances are a statement about the *unconditional* variances, but all the price relations are statements about *conditional* expectations and hence conditional variances. Kleidon (1986) made this point. The inequalities should be interpreted as

  $$\text{var}(p_t^*|I_t) \geq \text{var}(p_t|I_t)$$

  This is the relationship that should be tested, not the unconditional one.

  West (1988) showed how we could use a subset of the information set to generate testable implications on the variance of the price. Using similar notation to before, we can show that

  $$E\left[(p_t^* - E[p_t|Z_{t-1})^2\right] \geq E\left[(p_t^* - E[p_t|I_{t-1})^2\right]$$

  where Z_t is the econometricians information set, I_t the market's.

  This is a more interesting inequality in terms of looking at conditional relations.
- Conceptual problem: The constancy of the discount factor. We calculated the price as:

  $$p_t = \sum_{j=1}^{\infty} \beta^j E_t[d_{t+j}]$$

  Note that the factor β in this equation is a constant. However, there is no reason to expect that the discount factor is a constant. In the usual models this discount factor is a function of the marginal utilities of consumption, and therefore changing over time. Grossman and Shiller (1981) tries to take this into account, but adjusting for changes in the consumption growth is not enough to explain this. The problem is that the aggregate consumption growth series is hardly changing at all. We would need volatile consumption series for the volatility of consumption and hence marginal rate of substitution to influence the discount factors enough. However, the consumption series growth series is non-volatile. Hence, it doesn't help enough to explain the volatile stocks. We will return to this when we discuss the consumption based asset pricing theories.
- One point that Kleidon (1986) made was that we only have one observation of a particular time series, and there may be something weird about that one observation. This is not a valid criticism. If we have en ergodic time series, we will *eventually* observe all possible outcomes, and be able to infer all characteristics of the underlying process.
- Dividend smoothing. Recall what we know about dividends from corporate finance. It is often supposed that managers tend to smooth dividends in the sense that they are reluctant to change the level of dividends. This is an ingredient in most signalling models of asymmetric information. If this is the case, it may be that the change in the underlying fundamentals is not reflected in the fundamental value calculated using dividends. If this is the case, the small levels of volatility of dividends may not reflect true levels of volatility of the underlying fundamental.

However, trying to model this in a consistent way is problematic. We would have to account for the capital gain changes. But capital gains are only relevant when we have more than one owner of a stock. Note that sales between two agents is zero-sum for the whole economy. It is difficult to put capital gains into the usual representative agent framework, where we would have only one owner.

- Alternative way of testing the underlying relationship. We wanted to test the relation that

$$p_t = E[p_t^*|I_t] = p_t^* + \varepsilon_t$$

where ε_t is the estimation error. There is nothing about this relationship that says it has to be tested using variance bounds, an equivalent test would use the regression

$$p_t = a + bp_t^* + \epsilon_t$$

If we want to test whether $E[p_t] = E[p_t^*]$ we could simply test whether $a = 0$, and $b = 1$.

2.5 Explanations.

The evidence is that there is too much volatility in prices to be explained by a dividend based model with constant discount factors.

Some has taken this as evidence of irrational behaviour of the participants in the stock market. But as theorists, it is hard to accept exogenous irrationality as an explanation, since any irrationality should be self-destroying in the sense that if some people behave irrationally, "more" rational people should be able to use the irrationality to generate profits.

It seems more fruitful to try to accommodate the results of the variance bounds tests in a model with rational traders, and see how we have to modify assumptions to get models consistent with the observed data.

Some main trusts of the literature is

- "Bubbles" or "Fads:" Prices take swings away from the fundamental values.
- "Market Microstructure" literature, where we model interactions between traders in situations of incomplete or imperfect information. Most of these models rely on the existence of "noise traders" that have to trade securities for exogenous reasons.
- Modelling changing discount factors.

2.6 The Cochrane (1992) paper

Question: How do we do the type of tests done by Shiller in a fashion that is not subject to the econometric problems of that analysis.

Specification issues listed in critiques of Shiller

- Stationarity.

  To make statistical inferences, need stationary series. No reason to think prices & dividends are stationary. Use variables likely to be stationary. $\frac{P}{D}$, dividend growth, discount rate.

- Terminal prices.

  Biases by truncating at horizon. Avoided by looking at covariances.

- Time series restrictions.

  Need to allow for serial correlation in data.

- Mean of $\frac{P}{D}$ ratio and discount rate trade off. Need to look at this as well as variance bounds.

- Time varying discount rates.

Results: Find that changing discount rates can explain the variance inequalities, but can not tie the changes in discount rates to consumption, consumption is not variable enough. The changing discount rates have the same problems as the consumption-based asset pricing models, we can not find an explanation that is tied to a reasonable model of discount rate behaviour.

3 Bounding the Intertemporal marginal rate of substitution.

3.1 Introduction

Recall the pricing equation.

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

In a consumption based asset pricing model we will have $m_t = \frac{u'(c_{t+1})}{u'(c_t)}$. By parameterising $u(c_t)$, this was tested by Hansen and Singleton (1982).

But it is also possible to infer properties of m_t without making further assumptions. We can view this as a non-parametric approach, the properties of m_t identified will have to hold for all candidate m_t parameterisations.

3.2 Derivation of the bound with a risk free asset

Consider

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

Suppose we have a risk free asset $R_{f,t}$. Then

$$E_t[m_{t+1}R_{f,t}] = 1$$

Sincd $R_{f,t}$ is constant we can move it outside the expectation.

$$E_t[m_{t+1}]R_{f,t} = 1$$

Subtract the two to get the excess return $r_{i,t+1} = R_{i,t+1} - R_{f,t}$.

$$E_t[m_{t+1}R_{i,t+1}] - E_t[m_{t+1}]R_{f,t} = 1 - 1 = 0$$

$$E_t[m_{t+1}r_{i,t+1}] = 0$$

Using

$$0 = E_{t-1}[m_t r_{it}] = \text{cov}_{t-1}(m_t, r_{it}) + E_{t-1}[m_t]E_{t-1}[r_{it}]$$

we have

$$\text{cov}_{t-1}(m_t, r_{it}) = E_{t-1}[m_t]E_{t-1}[r_{it}]$$

Now use the fact that $\text{cov}(x, y) = \sigma(x)\sigma(y)\rho(x, y)$ to get:

$$\rho(m_t, r_{it})\sigma(m_t)\sigma(r_{it}) = E_{t-1}[m_t]E_{t-1}[r_{it}]$$

By the definition of correlation, $\rho > -1$. This implies that

$$-1\sigma(m_t)\sigma(r_{it}) \leq E_{t-1}[m_t]E_{t-1}[r_{it}]$$

$$\frac{\sigma(m_t)}{E[m_t]} \geq \frac{E[r_{it}]}{\sigma(r_{it})}$$

Since this will hold for any i, we get that

$$\frac{\sigma(m_t)}{E[m_t]} \geq \max_i \frac{E[r_{it}]}{\sigma(r_{it})}$$

Note that the expression in returns is a Sharpe Ratio.

This is one of the results in the Hansen and Jagannathan (1991) papers.

3.3 A similar bound without a risk free asset

Let us now use the more standard asset pricing setup in terms of notation. The price $\mathbf{q}$ of an asset is a product of the cash flows $\mathbf{x}$ and the stochastic discount factor y. We have that

$$\mathbf{q} = E[y\mathbf{x}]$$

We also know that

$$E[y\mathbf{x}] = E[y]E[\mathbf{x}] + \text{cov}(y, \mathbf{x})$$

Question: Is $y = \mathbf{x}'E[\mathbf{x}\mathbf{x}']^{-1}\mathbf{q}$ a candidate discount factor?

Regress the discount factor y on the observed asset returns $\mathbf{x}$.

$$y = a + \mathbf{x}\mathbf{b} + e$$

By definition,

$$\mathbf{b} = \text{cov}(\mathbf{x}, \mathbf{x})^{-1}\text{cov}(\mathbf{x}, y)$$

This can only be estimated if we know y.

But when y is the true stochastic discount factor, we know that

$$\mathbf{q} = E[y\mathbf{x}] = E[y]E[\mathbf{x}] + \text{cov}(\mathbf{x}, y)$$

or

$$\text{cov}(\mathbf{x}, y) = \mathbf{q} - E[y]E[\mathbf{x}]$$

Substitute for $\text{cov}(\mathbf{x}, y)$:

$$\mathbf{b} = \text{cov}(\mathbf{x}, \mathbf{x})^{-1}\left(\mathbf{q} - E[y]E[\mathbf{x}]\right)$$

Note, that the only unobservable factor here is the mean $E[y]$. All others are observable.

From

$$y = a + \mathbf{x}'\mathbf{b} + e$$

we get

$$\text{var}(y) = \text{var}(\mathbf{x}'\mathbf{b}) + \text{var}(e)$$

Implying that

$$\text{var}(y) \geq \text{var}(\mathbf{x}'\mathbf{b})$$

Hence, substituting for $\mathbf{b}$

$$\text{var}(y) \geq \text{var}\left(\mathbf{x}'\left\{\text{cov}(\mathbf{x}, \mathbf{x})^{-1}\left(\mathbf{q} - E[y]E[\mathbf{x}]\right)\right\}\right)$$

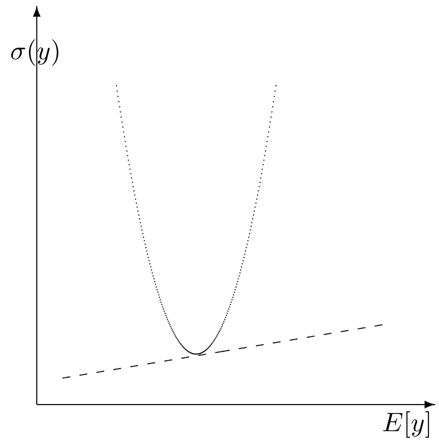
This is in the case where we don't have a risk free asset.

If we have a risk-free asset, the bound is as we showed earlier,

$$\frac{\sigma(y)}{E[y]} \geq \frac{E[r]}{\sigma(r)},$$

where r is the return on some risky asset.

The variance bounds will be shown like the following picture



The admissible area for the discount factors will be above the dashed line in the case of a risk free asset. The curve defines the area in the case of no risk free asset.

3.4 Implementation.

How do we go about implementing the variance bounds? The bounds are written in terms of expectations and variances. To implement, we need to estimate sample means and variances.

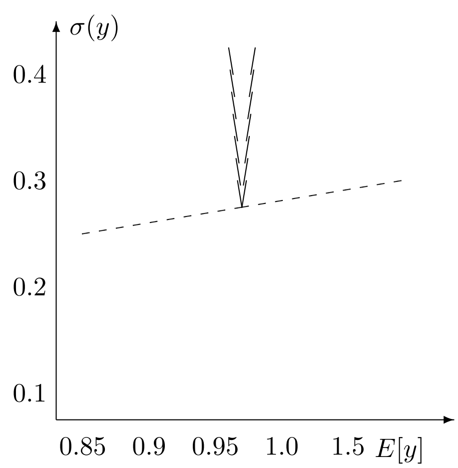
The pricing formulas are written in terms of *conditional* pricing functions. We can try to look at

- Conditional Gallant, Hansen, and Tauchen (1990) or
- Unconditional Hansen and Jagannathan (1991) variance bounds.

To use unconditional expectations we need to assume some kind of stationarity on the stochastic process generating returns. If we make these assumptions, we can estimate the expectation and variance from their sample counterparts.

3.5 Results.

Using sample averages on US data, the results are as shown in the figure below.



3.6 Equity Premium Puzzle

The Equity Premium puzzle of Mehra and Prescott (1985) is a famous result. Essentially, it says that the return on equity is too high to be justified in a representative agent setting. The necessary risk aversion of the utility function of the representative agent is way too high relative to what one estimate for individuals.

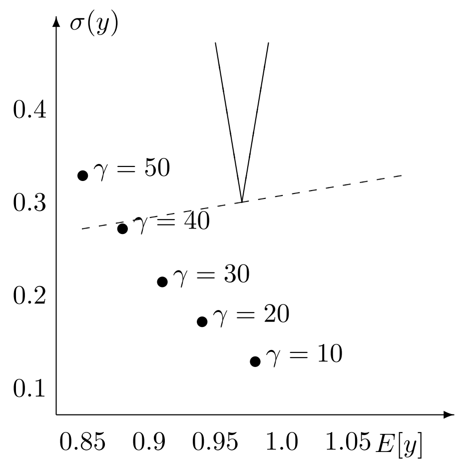
The Equity Premium puzzle of Mehra and Prescott (1985) can be illustrated in the imrs bound setting. In their setting the discount factor y is

$$y = \beta \frac{U(c_t + 1)}{U(c_t)}$$

and the utility function is

$$U(c) = \frac{1}{1-\gamma} c^{1-\gamma}$$

To estimate mean and standard deviation of this discount factor, we pick values of β and γ, calculate a time series of $\beta \frac{U'(c_t+1)}{U'(c_t)}$ and estimate its mean and standard deviation.



This is the Equity premium puzzle. To get into the admissible region we need a risk aversion coefficient γ of 40.

3.6.1 Sharpening the bound, impose positivity.

The original pricing equality is that

$$\pi(\mathbf{x}) = E[y\mathbf{x}]$$

We can sharpen this by imposing that the pricing operator must be positive:

$$\pi(\mathbf{x}) = E[\max(y, 0)\mathbf{x}]$$

Under no arbitrage, the stochastic discount factor must be positive. Imposing this will sharpen the variance bounds.

3.7 Problem: Sample point estimates, no testing.

Note that in the inequalities above, we only use point estimates of standard deviations and means in the inferences. In our specification of the equity premium puzzle, we do not take into account that there is sampling error in our estimates of means and standard deviations of the discount factors, on both the variance bounds, and the estimates given a utility function.

How to take this into account?

Use a test of overidentifying restrictions in a GMM type estimation.
What are the implications of the pricing relation?

$$\begin{aligned} q &= E[\mathbf{x}y] \\ &= E[\mathbf{x}(a + x'\mathbf{b} + e)] \\ &= E[\mathbf{x}a] + E[\mathbf{x}\mathbf{x}'\mathbf{b}] + E[\mathbf{x}e)] \\ &= E[\mathbf{x}(a + \mathbf{x}'\mathbf{b})] \\ &= E\left[\mathbf{x}\left[\begin{array}{c} a \\ \mathbf{x}'\mathbf{b} \end{array}\right]\right] \end{aligned}$$

Which implies

$$E\left[\mathbf{x}\,[1\ \mathbf{x}]]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] - q\right] = 0$$

We can go a bit further. What is the price today of an asset with unit payoff in all states?

$$q = E[y1] = E[y]$$

$E[y]$ is the inverse of the risk free interest rate if it exists.

$$q = \frac{1}{1 + r_f}$$

The moment condition is

$$E\left[1[1\ \mathbf{x}]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] - E[y]\right] = 0$$

Stack these two moment conditions.

$$E\left[\left[\begin{array}{c} 1 \\ \mathbf{x} \end{array}\right][1\ \mathbf{x}]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] - \left[\begin{array}{c} E[y] \\ \mathbf{q} \end{array}\right]\right] = 0$$

We further want a condition on the variance $\text{var}(a + \mathbf{x}'\mathbf{b})$.

$$E[(a + \mathbf{x}'\mathbf{b})^2] \leq E\left[y^2\right]$$

$$\begin{aligned} E[(a + \mathbf{x}'\mathbf{b})^2] &= E\left[[1\ \mathbf{x}']\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right][1\ \mathbf{x}]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right]\right] \\ &= E\left[[a\ \mathbf{b}']\left[\begin{array}{c} 1 \\ \mathbf{x} \end{array}\right][1\ \mathbf{x}']\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right]\right] \end{aligned}$$

Note that we can use the first moment condition here to find an expression for $E[(a + \mathbf{x}'\mathbf{b})^2]$:

$$E\left[\left[\begin{array}{c} 1 \\ \mathbf{x} \end{array}\right][1\ \mathbf{x}]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] - \left[\begin{array}{c} E[y] \\ \mathbf{q} \end{array}\right]\right] = 0$$

Premultiply with $\left[a\ \mathbf{b}'\right]$ and find

$$\begin{aligned} E[(a + \mathbf{x}'\mathbf{b})^2] &= E\left[[a\ \mathbf{b}']\left[\begin{array}{c} 1 \\ \mathbf{x} \end{array}\right][1\ \mathbf{x}]\left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right]\right] \\ &= E\left[[a\ \mathbf{b}']\left[\begin{array}{c} E[y] \\ \mathbf{q} \end{array}\right]\right] \end{aligned}$$

Hence, the variance inequality implies

$$E\left[[E[y] \;\; \mathbf{q}']\begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \leq E[y^2]\right]$$

If we have a candidate discount factor m, we are left with the following set of restrictions:

$$\begin{aligned} E\left[\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}[1 \; \mathbf{x}]\begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} - \begin{bmatrix} m \\ \mathbf{q} \end{bmatrix}\right] &= 0 \\ E\left[[m \;\; \mathbf{q}']\begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} - m^2\right] &\leq 0 \end{aligned}$$

This can be tested using GMM type estimation. Return to the standard model, with $U(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. The results from running these tests are in table 1 in the paper. We reject the null of this model being able to explain the data. Hence sampling variance can not explain the picture above.

Conclusion: The standard model with a representative consumer, and time-and-state-separable utility has problems in explaining asset returns.

Question: What is most fruitful to modify in the model in order to explain the asset returns? Some examples.

- Change the utility function. Constantinides (1990) assumes that the utility function is not time-separable, current utility depends on past consumption levels. The utility at time t depends on $c_t - \theta c_{t-1}$, instead of c_t only.
- Nonexpected utility. Epstein and Zin (1991) is the prime example of this. They relax the state-independence of the utility. They let your utility depend among other on "how you got where you are." In other words, you care about your resolution of uncertainty. See Epstein (1993) for a review of some of this large literature.
- Market Frictions. Restrictions on the moments where we still have aggregation, but there are frictions on the behaviour of the representative consumer, such as borrowing constraints.
- Disaggregation. Look at what happens when aggregation breaks down, there is more than one consumer.

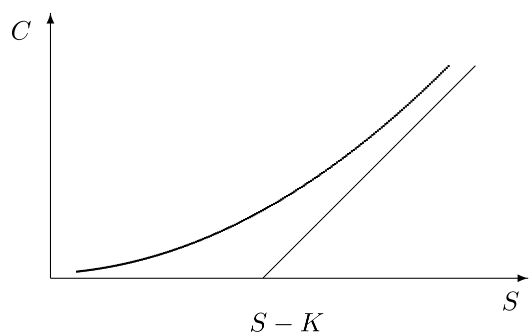
All of these will to some degree "help" in explaining the asset pricing puzzles we discuss, but none of them has been singled out as the "real" explanation.

4 Good-Deal Asset Pricing Bounds

Bounds involving the "pricing factor" m are not just for investigating the deep questions of macrofinancial asset pricing. They have many other uses.

In this section we look at an example application to option pricing, due to Cochrane and Saá-Requejo (1999), (see also the textbook treatment in (Cochrane, 2005, Ch 18)).

In option pricing, a call option is the present value of the future payoff $(X - K)^+$. When the ideal conditions of the Black Scholes Environment hold, we can calculate the exact Black Scholes price. as a function $c = c(S, K, (T - t), r, \sigma)$.

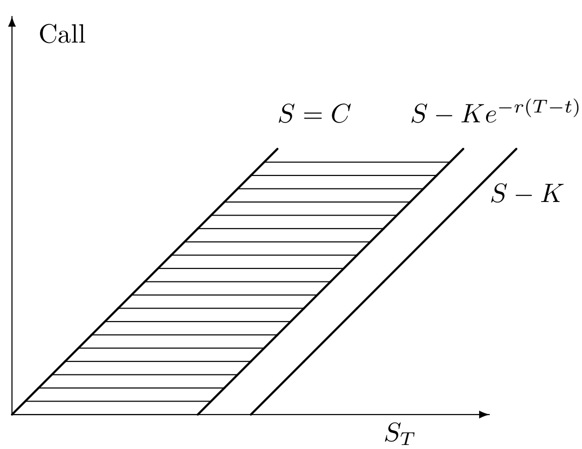


We know that there also exist a corresponding pricing factor m^* which would also price the call

$$c = E[m^*(S - K)^+]$$

However, the ideal conditions of the Black Scholes model (continous trading, no transaction costs) are not always present.

We can then go to "preference free" arbitrage arguments to put bounds on the option price, such as illustrated below.



But these bounds are so wide that they do not help us much if we need to set a price.

Can we do better?

The suggestion of Cochrane and Saá-Requejo (1999) is to look at candidate "pricing kernels" $m()$.

We know that there is some m that will price the option

$$c = E[m(S - K)^+]$$

Can we establish upper and lower bounds on c by restricing the set of feasible m?

An obvious way of doing that is to use prices $\mathbf{p}$ of other financial contracs (basis assets) with future payoffs $\mathbf{x}$

$$\mathbf{p} = E[m\mathbf{x}]$$

With a set of basis assets one sould search for the m that minimizes the program

$$\underline{C} = \min_{\{m\}} E\left[m(S-K)^+\right]$$

subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

A similar program could be used to establish an upper bound

$$\overline{C} = \max_{\{m\}} E\left[m(S-K)^+\right]$$

subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

How to make the bounds tighter?

For one thing we can add nonnegative state prices

$$m \geq 0$$

(This follows from the absense of arbitrage in complete markets, where a negative state price would generate infinite profits. In less complete markets it is still hard to think about getting money today for a possible future payoff...)

But, can also argue economically. (This is where the variance bound appears)

Cochrane and Saá-Requejo (1999) introduce the notion of "good deals", trading opportunities with very high Sharpe ratios. Such "good deals" will be traded away

If we can establish an upper bound h on the Sharpe Ratio

$$\frac{\left|R^h\right|}{\sigma(R^h)} = h$$

we can use this as an additional inequality constraint in the optimization programs.

$$h \geq \frac{\sigma(m)}{E[r_m]}$$

If there exists a risk free asset with (gross) returns R_f, will have

$$\text{price of unit payoff} = E[m \cdot 1] = E[m] = \frac{1}{R_f}$$

Hence

$$h \geq \frac{\sigma(m)}{\frac{1}{R_f}}$$

or

$$\frac{h}{R_f} \geq \sigma(m)$$

So we have a "variance bound" on the pricing kernel, which translates into "good deal" bounds

$$\underline{C} = \min_{\{m\}} E\left[m(S-K)^+\right]$$

$$\overline{C} = \max_{\{m\}} E\left[m(S-K)^+\right]$$

both subject to

$$\mathbf{p} = E[m\mathbf{x}]$$

$$m > 0$$

$$\frac{h}{R_f} \geq \sigma(m)$$

In their paper Cochrane and Saá-Requejo (1999) show the following example application

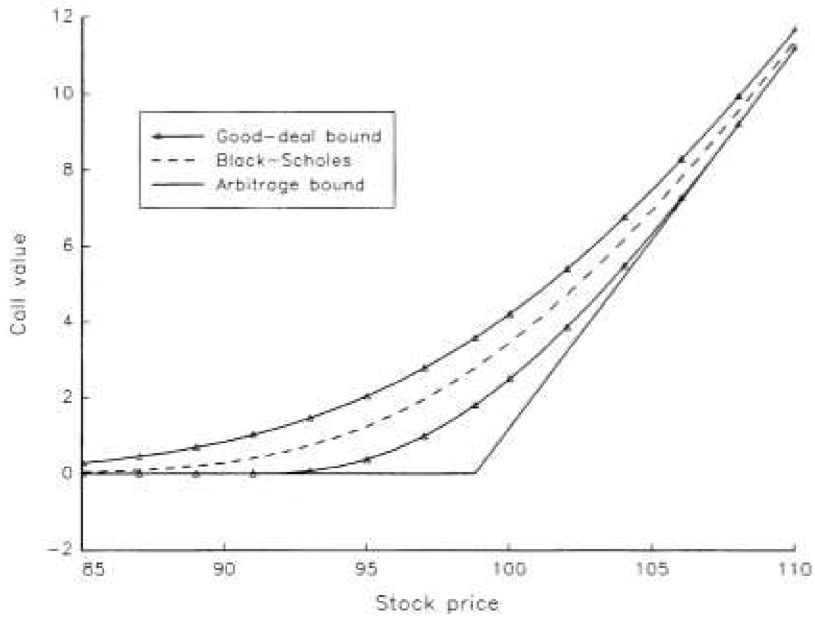


FIG. 1.—Option price bounds as a function of stock price. Options have three months to expiration and strike price $K = \$100$. The bounds assume no trading until expiration and a discount factor volatility bound $h = 1.0$ corresponding to twice the market Sharpe ratio. The stock is lognormally distributed with parameters calibrated to an index option.

4.1 References

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