SDF based asset pricing

Bernt Arne Ødegaard

24 November 2021

Contents

1 General overview of asset pricing testing.

The purpose of this section is to give an overview of a number of asset pricing models, their testing, and relation to each others.[1] Consider what is typically called the *canonical asset pricing equation*. Most of the models we will look at can be viewed as special cases of this.

$$E_t[m_{t+1}R_{it+1}] = 1 \tag{1}$$

Here $R_{i,t+1}$ is the gross return, and m_t is a random variable. The exact nature of m_t will depend on the nature of our asset pricing model.

$E_t[\cdot]$ is shorthand for the *conditional expectation* given a time t information set. This would be written more correctly as $E[\cdot|\Omega_t]$, where Ω_t is the *market-wide* information set.[2]

This equation is the outcome of a number of models, and m_t has many names, depending on the model. Examples include the intertemporal marginal rate of substitution, a stochastic discount factor, and an equivalent Martingale measure.

1.1 Pricing operators

Let me now give a quick reasoning for where this equation is coming from.

Notation. The equation in return form is:

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

[1] Some references to this material: Ferson (1995), Cochrane's book

[2] You may also want to recall the *Law of iterated expectations*: $E[X] = E\left[E[X|Y]\right]$, for random variable X and Y, which is heavily used in econometric analysis. In the shorthand form used above, this can be written

$$E_t[y_{t+2}] = E_t\left[E_{t+1}[y_{t+2}]\right]$$

Since

$$R_{i,t+1} = \frac{P_{i,t+1}}{P_{i,t}}$$

We can rewrite

$$E_t \left[m_{t+1} \frac{P_{i,t+1}}{P_{i,t}} \right] = 1$$

implying

$$P_{i,t} = E_t \left[m_{t+1} P_{i,t+1} \right]$$

Let us now map this notation to the the more common asset prcing one.

The *future* payoffs for asset i:

$$P_{i,t+1} = x_i$$

Stack these:

$$\begin{bmatrix} P_{1,t+1} \\ \vdots \\ P_{n,t+1} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

or

$$\mathbf{P}_{t+1} = \mathbf{x}$$

The interpretation is that $\mathbf{x}$ is the vector of future payoffs.

Further, current payoffs

$$\mathbf{P}_t = \mathbf{q}$$

and the factor

$$m_{t+1} = y$$

We are interested in the price today of the vector $\mathbf{x}$ of future payoffs. This is the *pricing functional* $\pi(\cdot)$ that maps future payoffs into current prices. The prices today of the future payoffs $\mathbf{x}$ is $\mathbf{q}$:

$$\mathbf{q} = \pi(\mathbf{x})$$

Since $\pi(\cdot)$ represent current prices of claims to future payoffs, we can say something about it.[3] For obvious no-arbitrage reasons, it makes sense to impose *value-additivity*:

$$\pi(\omega_1 x_1 + \omega_2 x_2) = \omega_1 \pi(x_1) + \omega_2 \pi(x_2)$$

and continuity, very small payoffs have small prices. These are sufficient assumptions to restrict $\pi(\cdot)$ to be a *linear functional* on the space of future payoffs. This has a well-defined meaning in Hilbert Space theory, but we dont go into details here.

$$\mathbf{q} = \pi(\mathbf{x})$$

If $\mathbf{c}$ is a portfolio of assets, linearity implies that

$$\mathbf{cq} = \pi(\mathbf{cx})$$

Consider now this linear functional $\pi(\cdot)$. Suppose we want to represent this with some object, such as a function of a portfolio of payoffs, that is, we want to represent prices with an object we can relate to.

It can be shown that any pricing functional $\pi(\cdot)$ can be represented by a random variable y as:[4]

[3] See e.g. Hansen and Richard (1987)

[4] This uses the Riesz representation theorem on Hilbert Spaces. We need that

- The set of payoffs is a linear space H.
- The conditional expectation defines an inner product on this linear space. If x, y are in the space H, the conditional expectation $E[xy]$ is an inner product.
- The set of payoffs with the inner product of conditional expectation is a Hilbert Space.

The Riesz Representation Theorem says that for a bounded linear functional f on the space H with inner product $(\cdot, \cdot)$, there exist an unique element x_0 in H such that

$$f(x) = (x, x_0)$$

If the conditional expectation is the inner product, for any linear functional $f(\mathbf{x})$, there exist a y such that

$$f(\mathbf{x}) = E[y\mathbf{x}].$$

$$\mathbf{q} = \pi(\mathbf{x}) = E[y\mathbf{x}]$$

That is, there is some random variable y that can be used to price all payoffs $\mathbf{x}$.

This variable y is the *stochastic discount factor*.

2 Present value relationship.

Let us look at one implication of (1). It can be used as a justification of the **present value model**:

$$\begin{aligned}
P_t &= E_t[m_{t+1}(d_{t+1} + p_{t+1})] \\
&= E_t[m_{t+1}d_{t+1} + m_{t+1}E_{t+1}[m_{t+2}(d_{t+2} + p_{t+2})]] \\
&= E_t[m_{t+1}d_{t+1} + m_{t+1}m_{t+2}(d_{t+2} + p_{t+2})] \\
&= E_t[m_{t+1}d_{t+1} + m_{t+1}m_{t+2}(d_{t+2} + E_{t+2}[d_{t+3} + p_{t+3}])] \\
&\vdots \\
&= E_t\left[\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} m_{t+j}\right) d_{t+i}\right]
\end{aligned}$$

That is, the price of any stream of cash flows is its discounted present value. Note that this assumes that the limit of $\left(\prod_{j=1}^{i} m_{t+j}\right)$ as $i \to \infty$, is finite.

3 The Lucas (1978) type analysis.

We go through the derivation of the *canonical asset pricing equation* in one special case.

The setting is a general equilibrium model, where we posit the existence of a *representative consumer* who is maximising his (or hers) utility of future consumption.

Let c_t be the consumption in period t. There is only one asset in the economy, with price p_t and paying dividends of d_t in period t. Let q_t be the agents holdings (quantity) of the asset at the beginning of period t. The consumer is assumed to have wage income of w_t.

It should be easy to verify that the agents *budget constraint* is

$$c_t + p_t q_t \leq (p_t + d_t)q_{t-1} + w_t$$

The consumer is assumed to maximise his lifetime expected utility

$$E_0\left[\sum_{t=1}^{\infty} \beta^t u(c_t)\right]$$

where β is a discount factor.

We will close this model by noting that in equilibrium, the demand of assets is equal to the supply, and we have only one agent, $q_t = q_{t+1}\ \forall\ t$.

The problem we want to solve is then

$$\max_{\{c_t, q_t\}} E_0\left[\sum_{t=1}^{\infty} \beta^t u(c_t)\right]$$

subject to

$$c_t + p_t q_t \leq (p_t + d_t)q_{t-1} + w_t$$

for $t = 0, 1, 2, \cdots$.

This problem can be solved in a number of ways, the most standard being by dynamic programming. But let us look at what may be the simplest, doing the optimisation directly by forming a Lagrangian:[5]

$$L = E_0 \left[\sum_{t=1}^{\infty} \beta^t u(c_t) - \sum_{t=1}^{\infty} \lambda_t \left(c_t + p_t q_t - (p_t + d_t) q_{t-1} - w_t \right) \right]$$

Take derivatives wrt c_r and q_r we get

$$\begin{aligned} \frac{\partial L}{\partial c_r} &= E_0 \left[\beta^r u'(c_r) \right] - \lambda_r &= 0 \\ \frac{\partial L}{\partial q_r} &= -\lambda_r p_r + \lambda_{r+1}(p_{r+1} + d_{r+1}) &= 0 \end{aligned}$$

Use the first equation to substitute in the second, and we get a condition for optimality that will need to hold for any c_t.

$$E_t[\beta^t u'(c_t) p_t] = E_t \left[\beta^{t+1} u'(c_{t+1})(d_{t+1} + p_{t+1}) \right]$$

or

$$E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{(p_{t+1} + d_{t+1})}{p_t} \right] = 1$$

This is usually called the *Euler equation* in this type of model.

4 Beta-pricing relations.

We can also use our fundamental equation to look at *beta-pricing* style relations. Let us first write (1) in standard return form by subtracting 1 from the gross return:

$$r_{it} = R_{it} - 1$$

which gives

$$E_t[m_{t+1} r_{i,t+1}] = 0 \tag{2}$$

Recall the definition of covariance.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

Rewrite this for our variables:

$$\text{cov}_{t-1}(m_t, r_{it}) = E_{t-1}[m_t r_t] - E_{t-1}[m_t] E_{t-1}[r_{it}]$$

Solve for $E_{t-1}[r_{it}]$:

$$\begin{aligned} & \text{cov}_{t-1}(m_t, r_{it}) + E_{t-1}[m_t] E_{t-1}[r_{it}] = E_{t-1}[m_t r_{it}] = 0 \\ \Rightarrow \quad & \text{cov}_{t-1}(m_t, r_{it}) + E_{t-1}[m_t] E_{t-1}[r_{it}] = 0 \\ \Rightarrow \quad & -\text{cov}_{t-1}(m_t, r_{it}) = E_{t-1}[m_t] E_{t-1}[r_{it}] \\ \Rightarrow \quad & \frac{-\text{cov}_{t-1}(m_t, r_{it})}{E_{t-1}[m_t]} = E_{t-1}[r_{it}] \end{aligned}$$

or

$$E_{t-1}[r_{it}] = \frac{\text{cov}_{t-1}(-m_t, r_{it})}{E_{t-1}[m_t]}$$

This is a relationship between the return on any asset with its covariance with the pricing variable m_t. In the case of a consumption-based model such as the one studied above, the return on asset i is a function of the asset's covariance with consumption.

[5] In this we are not using the usual dynamic formulation. We need some additional restrictions on the solution for this approach to be correct in general.

5 Special case, CAPM – style relations.

In the previous we found the return on any asset i as a function of its covariance with the variable m_t.

$$E_{t-1}[r_{it}] = \frac{\text{cov}_{t-1}(-m_t, r_{it})}{E_{t-1}[m_t]}$$

We now want to show how our familiar asset pricing model the CAPM can be shown to be a special case of this.

Remember that the CAPM specifies a relationship with the market portfolio. Let us first consider the return on any portfolio p, r_{pt}, (not necessarily the market portfolio), with $\text{cov}_{t-1}(r_{pt}, m_t) \neq 0$.

From the definition of covariance,

$$\begin{aligned}
\text{cov}_{t-1}(r_{pt}, -m_t) &= E_{t-1}[-m_t r_{pt}] - E_{t-1}[r_{pt}]E_{t-1}[-m_t] \\
&= -0 + E_{t-1}[r_{pt}]E_{t-1}[m_t] \\
&= E_{t-1}[r_{pt}]E_{t-1}[m_t]
\end{aligned}$$

Hence

$$E_{t-1}[m_t] = \frac{\text{cov}_{t-1}(-m_t, r_{pt})}{E_{t-1}[r_{pt}]}$$

Now substitute for $E_{t-1}[m_t]$ in the equation for $E_{t-1}[r_{it}]$.

$$E_{t-1}[r_{it}] = \frac{\text{cov}_{t-1}(-m_t, r_{it})}{E_{t-1}[m_t]}$$

giving

$$\begin{aligned}
E_{t-1}[r_{it}] &= \frac{\text{cov}_{t-1}(-m_t, r_{it})}{\frac{\text{cov}_{t-1}(-m_t, r_{pt})}{E_{t-1}[r_{pt}]}} \\
&= \frac{\text{cov}_{t-1}(-m_t, r_{it})}{\text{cov}_{t-1}(-m_t, r_{pt})} E_{t-1}[r_{pt}]
\end{aligned}$$

We now have a relationship where the portfolio return appear. Let us next try to get rid of the pricing variable m_t.

Consider replacing m_t with an *estimate*, a function of the returns of the assets in the portfolio. We use a linear regression on the vector

$$R_t = \begin{bmatrix} R_{1,t} \\ \vdots \\ R_{n,t} \end{bmatrix}$$

of individual asset returns

$$m_t = \omega_t R_t + \varepsilon_t$$

By a known result,[6] there is always a vector ω_t[7] such that

$$E_{t-1}[\varepsilon_t' R_t] = 0$$

[6]This result is known as the *projection theorem*, but since this is usually formulated in Hilbert Spaces, it goes beyond this course. Essentially, the result says that error from use of the best estimator (best in the sense of minimising distance) will always be orthogonal to the information used in making the estimate.

For the specially interested, here is the formulation of the *Classical Projection Theorem*: Let H be a Hilbert space and M a closed subspace of H. Corresponding to any vector $x \in H$, there is an unique vector $m_0 \in M$ such that $\|x - m_0\|$ for all $m \in M$. Furthermore, a necessary and sufficient condition that m_0 be the unique minimising vector is that $x - m_0$ is orthogonal to M. See Luenberger (1969).

[7]In this case we can actually calculate this vector ω_t: We know

$$E_t[m_{t+1} R_{i,t+1}] = 1 \ \ \forall\, i$$

or

$$E_t[m_{t+1} R_{t+1}] = \mathbf{1}$$

or equivalently

$$\text{cov}_{t-1}(R_t, \varepsilon_t) = 0$$

The regression coefficients ω_t of this regression are not guaranteed to sum to one, but we fix that by normalising the weights with the sum: $\mathbf{1}'\omega_t$, where $\mathbf{1}$ is the unit vector. We then have found portfolio weights $\frac{1}{\mathbf{1}'\omega_t}\omega_t$.

Then the return on the portfolio p is

$$R_{pt} = \frac{1}{\mathbf{1}'\omega_t}\omega_t' R_t$$

Also note that we can rewrite m_t as

$$m_t = \mathbf{1}'\omega_t R_{pt} + \varepsilon_t$$

Hence

$$\begin{aligned}
\text{cov}_{t-1}(-m_t, r_{it}) &= \text{cov}_{t-1}(-(\omega_t' R_t + \varepsilon), r_{it}) \\
&= \text{cov}_{t-1}(-\omega_t' R_t, r_{it}) + \text{cov}_{t-1}(\varepsilon_t, r_{it}) \\
&= -\mathbf{1}'\omega_t \text{cov}_{t-1}(R_{pt}, r_{it}) + 0 \\
&= -\mathbf{1}'\omega_t \text{cov}_{t-1}(R_{pt}, r_{it})
\end{aligned}$$

Use this to get

$$\begin{aligned}
E_{t-1}[r_{it}] &= \frac{\text{cov}_{t-1}(-m_t, r_{it})}{\text{cov}_{t-1}(-m_t, r_{pt})} E_{t-1}[r_{pt}] \\
&= \frac{-\mathbf{1}'\omega_t \text{cov}_{t-1}(R_{pt}, r_{i,})}{-\mathbf{1}'\omega_t \text{cov}_{t-1}(R_{pt}, r_{pt})} E_{t-1}[r_{pt}] \\
&= \frac{\text{cov}_{t-1}(r_{pt}, r_{it})}{\text{cov}_{t-1}(r_{pt}, r_{pt})} E_{t-1}[r_{pt}] \\
&= \frac{\text{cov}_{t-1}(r_{pt}, r_{it})}{\text{var}_{t-1}(r_{pt})} E_{t-1}[r_{pt}]
\end{aligned}$$

Finally, let us posit the existence of some asset z with return R_{zt}, and with $\text{cov}_{t-1}(R_{zt}, R_{pt}) = 0$. (usually called the "zero-beta" asset.)

We can then write

$$E_{t-1}[r_{it} - r_{zt}] = \frac{\text{cov}_{t-1}(r_{pt}, r_{it})}{\text{var}_{t-1}(r_{pt})} E_{t-1}[r_{pt} - r_{zt}]$$

If there is a risk free rate r_{ft}, by definition it has $\text{cov}_{t-1}(r_{pt}, r_{ft}) = 0$, and we get the CAPM in its usual form

$$E_{t-1}[r_{it}] - r_{ft} = \frac{\text{cov}_{t-1}(r_{pt}, r_{it})}{\text{var}_{t-1}(r_{pt})} (E_{t-1}[r_{pt}] - r_{ft})$$

Note that this is a *conditional* version of the CAPM, it holds given the current information set.

Substitute for m_{t+1}:

$$E_t[(\omega_{t+1} R_{t+1}) R_{t+1}] = \mathbf{1}$$

solve for ω_{t+1}:

$$\omega_{t+1} E_t[R_{t+1} R_{t+1}] = \mathbf{1}$$

and

$$\omega_{t+1} = (E_t[R_{t+1} R_{t+1}])^{-1}\mathbf{1}$$

6 Factor models, APT

By some more work, we can also get an APT-style relation in asset returns,

$$E_t[R_{i,t+1}] = \lambda_{0,t} + \sum_{j=1}^{K} b_{ijt} \frac{\text{cov}_t(F_{j,t+1}, -m_{t+1})}{E_t[m_{t+1}]}$$

as a special case of our generic relation.

The problem with the APT is that it is a relationship that holds for some "factors," but we do not know what the factors are.

There are two main methods used in estimation of the APT.

1. Estimate the factors from the data, using one of
   (a) Factor analysis.
   (b) Principal components analysis.
2. Prespecify the factors as economic variables we believe may influence asset returns.

7 Use of conditioning information.

The above shows how a large number of the models we know can be viewed as special cases of a relation

$$E_t[r_{t+1}m_{t+1}] = 0$$

Note that this formula is in the form of the *conditional* expectation.

The ability to use *conditioning information* in a meaningful way is one of the major breakthroughs in current research in empirical asset pricing.

In this class we will see how it is done in particular models, and how recent research differs from the classical tests.

Let me note a couple of ways to use conditioning information

- Use of variables in the information set as instruments in the estimation.
- Try to model the conditional expectations directly (latent variables)

8 Characterising m_t directly.

Usually, we do estimation in the context of particular *asset pricing model*. In the context of the equation

$$E_t[m_{t+1}R_{i,t+1}] = 1$$

this means putting some structure on m_t.

Some examples:

- the consumption based asset pricing model, where $m_t = \frac{u'(c_{t+1})}{u'(c_t)}$
- the CAPM, where we transformed this into a relationship with a *reference portfolio*.

Alternatively: write $m_t = f($ "factors") (in the factor analysis spirit), such as

$$m_t = 1 + ber_{m,t}$$

This approach gives us another way of asking what "pervasive" factors affects the crossection of asset returns.

Exercise 1.

The Stochastic Discount factor approach to asset pricing results in the following expression for pricing any excess return:

$$E[m_t er_{it}] = 0$$

Consider an empirical implementation of this where we write the pricing variable m as a function of a set of prespecified factors f:

$$m_t = 1 + bf_t$$

Consider the case of the one factor model $f = 1 + ber_m$, where the only explanatory factor is the return on a broad based market index.

Implement this approach on the set of 5 size sorted portfolios provided by Ken French. Use data 1926–2012.

Is the market a relevant pricing factor?

Solution to Exercise 1.

Reading data

```
source("read_size_portfolios.R")
source("read_pricing_factors.R")
eRi <- FFSize5EW - RF
data  <- merge(eRi,RMRF,all=FALSE)
summary(data)
eRi    <- as.matrix(data[,1:5])
eRm   <- as.vector(data[,6])
```

The specification of the GMM estimation:

```
X     <- cbind(eRi,eRm)
g1     <- function (parms,X) {
  b <- parms[1];
  f <- as.vector(X[,6])
  m <- 1 + b * f
  e <- m * X[,1:5]
  return (e);
}
```

Running the GMM analysis

```
t0   <- c(0.1);
res <-  gmm(g1,X,t0,method="Brent",lower=-10,upper=10)
summary(res)
```

Results

```
> summary(data)
     Index           Lo20              Qnt2               Qnt3
 Min.   :1926   Min.   :-32.010   Min.   :-31.9600   Min.   :-31.3100
 1st Qu.:1948   1st Qu.: -3.110   1st Qu.: -2.8650   1st Qu.: -2.5550
 Median :1970   Median :  0.960   Median :  1.1700   Median :  1.2000
 Mean   :1970   Mean   :  1.373   Mean   :  0.9704   Mean   :  0.8869
 3rd Qu.:1991   3rd Qu.:  4.695   3rd Qu.:  4.5350   3rd Qu.:  4.4050
 Max.   :2013   Max.   :110.670   Max.   : 81.1900   Max.   : 56.8400
      Qnt4             Hi20              RMRF
 Min.   :-29.760   Min.   :-30.100   Min.   :-28.980
 1st Qu.: -2.470   1st Qu.: -2.195   1st Qu.: -2.105
 Median :  1.160   Median :  0.930   Median :  1.010
 Mean   :  0.787   Mean   :  0.655   Mean   :  0.628
 3rd Qu.:  4.125   3rd Qu.:  3.640   3rd Qu.:  3.655
 Max.   : 50.010   Max.   : 41.790   Max.   : 37.770
```

GMM results

```
Call:
gmm(g = g1, x = X, t0 = t0, method = "Brent", lower = -10, upper = 10)

Method:  twoStep
Kernel:  Quadratic Spectral(with bw =  3.56894 )
```

```
Coefficients:
          Estimate    Std. Error  t value     Pr(>|t|)
Theta[1]  -0.0199775   0.0060763  -3.2877763   0.0010098

J-Test: degrees of freedom is 4
                J-test     P-value
Test E(g)=0:    14.606060   0.005592

Initial values of the coefficients
  Theta[1]
-0.0253752

#############
Information related to the numerical optimization
Convergence code =  0
Function eval. =  NA
Gradian eval. =  NA
```

	Model 1
Theta[1]	-0.02^{***}
	(0.01)
Criterion function	1411.21
Num. obs.	1035

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Exercise 2.

The Stochastic Discount factor approach to asset pricing results in the following expression for pricing any excess return:

$$E[m_t er_{it}] = 0$$

Consider an empirical implementation of this where we write the pricing variable m as a function of a set of prespecified factors f:

$$m_t = 1 + bf_t$$

Consider the case of the three factor model $f = 1 + b_1 er_m + b_2 SMB + b_3 HML$, where the explanatory factors are the return on a broad based market index, and the two Fama French factors SMB and HML.

Implement this approach on the set of 5 size sorted portfolios provided by Ken French. Use data 1926–2012.

Which are the relevant pricing factors?

Solution to Exercise 2.

Organizing the data

```
library(gmm)
source("read_size_portfolios.R")
source("read_pricing_factors.R")

eRi <- FFSize5EW - RF
data  <- merge(eRi,RMRF,SMB,HML,all=FALSE)
summary(data)
eRi    <- as.matrix(data[,1:5])
eRm   <- as.vector(data$RMRF)
smb <- as.vector(data$SMB)
hml <- as.vector(data$HML)
```

The GMM specification

```
X     <- cbind(eRi,eRm,smb,hml)
g3     <- function (parms,X) {
```

```
  b1 <- parms[1];
  b2 <- parms[2];
  b3 <- parms[3];
  erm <- as.vector(X[,6])
  smb <- as.vector(X[,7])
  hml <- as.vector(X[,8])
  m <- 1 + b1 * erm + b2*smb + b3*hml
  e <- m * X[,1:5]
  return (e);
}
```

Data, overview

```
> summary(data)
     Index          Lo20              Qnt2               Qnt3
 Min.   :1926   Min.   :-32.010   Min.   :-31.9600   Min.   :-31.3100
 1st Qu.:1948   1st Qu.: -3.110   1st Qu.: -2.8650   1st Qu.: -2.5550
 Median :1970   Median :  0.960   Median :  1.1700   Median :  1.2000
 Mean   :1970   Mean   :  1.373   Mean   :  0.9704   Mean   :  0.8869
 3rd Qu.:1991   3rd Qu.:  4.695   3rd Qu.:  4.5350   3rd Qu.:  4.4050
 Max.   :2013   Max.   :110.670   Max.   : 81.1900   Max.   : 56.8400
      Qnt4             Hi20             RMRF              SMB
 Min.   :-29.760   Min.   :-30.100   Min.   :-28.980   Min.   :-16.3900
 1st Qu.: -2.470   1st Qu.: -2.195   1st Qu.: -2.105   1st Qu.: -1.5200
 Median :  1.160   Median :  0.930   Median :  1.010   Median :  0.0500
 Mean   :  0.787   Mean   :  0.655   Mean   :  0.628   Mean   :  0.2352
 3rd Qu.:  4.125   3rd Qu.:  3.640   3rd Qu.:  3.655   3rd Qu.:  1.7750
 Max.   : 50.010   Max.   : 41.790   Max.   : 37.770   Max.   : 39.0400
      HML
 Min.   :-13.450
 1st Qu.: -1.295
 Median :  0.220
 Mean   :  0.382
 3rd Qu.:  1.745
 Max.   : 35.480
```

Running the GMM analysis

```
> t0   <- c(1.0,0,0);
> res <-  gmm(g3,X,t0)
> summary(res)

Call:
gmm(g = g3, x = X, t0 = t0)


Method:  twoStep

Kernel:  Quadratic Spectral(with bw =  4.83032 )

Coefficients:
          Estimate    Std. Error  t value     Pr(>|t|)
Theta[1]  -0.0133913   0.0086600  -1.5463407   0.1220223
Theta[2]   0.0131771   0.0145060   0.9083904   0.3636720
Theta[3]  -0.0778065   0.0264944  -2.9367123   0.0033171

J-Test: degrees of freedom is 2
                J-test   P-value
Test E(g)=0:    3.51806  0.17221

Initial values of the coefficients
```

```
    Theta[1]     Theta[2]     Theta[3]
-0.009775731  0.011717385 -0.076275222

#############
Information related to the numerical optimization
Convergence code =  0
Function eval. =  100
Gradian eval. =  NA
```

	Model 1
Theta[1]	-0.01
	(0.01)
Theta[2]	0.01
	(0.01)
Theta[3]	-0.08^{***}
	(0.03)
Criterion function	339.91
Num. obs.	1035

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Exercise 3.
Using the moment condition

$$E[m_t er_{it}] = 0$$

where

$$m_t = 1 + bf_t$$

Using data for 1980-2012, apply this to the one factor model $f = 1 + b_1 er_m$ and apply it to the set of ten size portfolios at the OSE.

Is er_m a significant determinant for the crossection?

Does it seem sufficient?

Solution to Exercise 3.
Reading the data

```
# estimate m=1+b*f in crossection
library(zoo)
library(texreg)
Rets <- read.zoo("../../data/equity_size_portfolios_monthly_ew.txt",
                 header=TRUE,sep=";",format="%Y%m%d")
Rf <- read.zoo("../../data/NIBOR_monthly.txt",
               header=TRUE,sep=";",format="%Y%m%d")
Rm <- read.zoo("../../data/market_portfolios_monthly.txt",
               header=TRUE,sep=";",format="%Y%m%d")
eRmew <- Rm$EW - lag(Rf,-1)
eR  <- Rets - lag(Rf,-1)
# take intersection to align data
data  <- merge(eR,eRmew,all=FALSE)
er  <- as.matrix(data[,1:10])
erm <- as.vector(data[,11])
```

The GMM specification of the moment conditions

```
X <- cbind(er,erm)
g <- function (parms,X) {
  b <- parms[1];
  f <- as.vector(X[,11])
  m <- 1 + b * f
  e <- m * X[,1:10]
  return (e);
}
```

Results

```
gmm(g = g, x = X, t0 = t0, method = "Brent", lower = -10, upper = 10)

Method:  twoStep
Kernel:  Quadratic Spectral(with bw =  3.60168 )

Coefficients:
          Estimate     Std. Error   t value      Pr(>|t|)
Theta[1]  -4.3783e+00   1.0679e+00  -4.1000e+00   4.1321e-05

J-Test: degrees of freedom is 9
                J-test      P-value
Test E(g)=0:    3.2419e+01  1.6849e-04

Initial values of the coefficients
 Theta[1]
-3.064137

#############
Information related to the numerical optimization
Convergence code =  0
Function eval. =  NA
Gradian eval. =  NA
```

To answer the two questions:

- the p value of the coefficient is used to answer the first, the market is a significant determinant.
- the p value of the J test is used to answer the second. Since we reject the J test, we do not find the one factor to be sufficient.

	Model 1
Theta[1]	-4.38^{***}
	(1.07)
Criterion function	8207.37
Num. obs.	395

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Exercise 4.

Using the moment condition

$$E[m_t er_{it}] = 0$$

where

$$m_t = 1 + bf_t$$

Using data for 1980-2012, apply this to the three factor model $f = 1 + b_1 er_m + b_2 SMB + b_3 HML$ and apply it to the set of ten size portfolios at the OSE.

Are the three factors significant determinants for the crossection?

Do they seem sufficient?

Solution to Exercise 4.

Reading the data

```
# estimate m=1+b*f in crossection
library(zoo)
library(texreg)
Rets <- read.zoo("../../data/equity_size_portfolios_monthly_ew.txt",
                 header=TRUE,sep=";",format="%Y%m%d")
Rf <- read.zoo("../../data/NIBOR_monthly.txt",
               header=TRUE,sep=";",format="%Y%m%d")
Rm <- read.zoo("../../data/market_portfolios_monthly.txt",
```

```
                header=TRUE,sep=";",format="%Y%m%d")
FF <- read.zoo("../../data/pricing_factors_monthly.txt",
                header=TRUE,sep=";",format="%Y%m%d")
eRmew <- Rm$EW - lag(Rf,-1)
eR  <- Rets - lag(Rf,-1)
data  <- merge(eR,eRmew,na.omit(FF$SMB),na.omit(FF$HML),all=FALSE)
er  <- as.matrix(data[,1:10])
erm <-as.matrix(data[,11])
SMB <- as.matrix(data[,12])
HML <- as.matrix(data[,13])
```

Doing the GMM

```
X <- cbind(er,erm,SMB,HML)
g <- function (parms,X) {
  b1 <- parms[1]
  b2 <- parms[2]
  b3 <- parms[3]
  m <- 1 + b1 * X[,11] + b2 * X[,12] + b3 * X[,13]
  e <- m * X[,1:10]
  return (e);
}
```

Results

```
> t0 <- c(-1,-1,-1)
> res <- gmm(g,X,t0)
> summary(res)

Call:
gmm(g = g, x = X, t0 = t0)

Method:  twoStep
Kernel:  Quadratic Spectral(with bw =  3.24629 )

Coefficients:
          Estimate    Std. Error  t value     Pr(>|t|)
Theta[1]  -3.6594378   1.1744252  -3.1159395   0.0018336
Theta[2]  -4.5064637   1.4152014  -3.1843268   0.0014509
Theta[3]  -7.4386994   2.9039257  -2.5616011   0.0104191

J-Test: degrees of freedom is 7
                J-test      P-value
Test E(g)=0:    22.8291039   0.0018254

Initial values of the coefficients
 Theta[1]  Theta[2]  Theta[3]
-2.655046 -4.971403 -8.739352

#############
Information related to the numerical optimization
Convergence code =  0
Function eval. =  140
Gradian eval. =  NA
```

Here see that all three pricing factors are significant, so they are influencing the crossection. We also reject that the model is sufficient, the J statistic is significant.

	Model 1
Theta[1]	-3.66^{***}
	(1.17)
Theta[2]	-4.51^{***}
	(1.42)
Theta[3]	-7.44^{**}
	(2.90)
Criterion function	6039.45
Num. obs.	378

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

8.1 Bounds on the stochastic discount factor

An alternative: Infer properties of m_t without making further assumptions.

Since

$$\begin{aligned} 0 &= E_{t-1}[m_t r_{it}] \\ &= \text{cov}_{t-1}(m_t, r_{it}) + E_{t-1}[m_t]E_{t-1}[r_{it}], \end{aligned}$$

we have

$$\text{cov}_{t-1}(m_t, r_{it}) = E_{t-1}[m_t]E_{t-1}[r_{it}]$$

Now use the fact that $\text{cov}(x, y) = \sigma(x)\sigma(y)\rho(x, y)$ to get:

$$\rho(m_t, r_{it})\sigma(m_t)\sigma(r_{it}) = E_{t-1}[m_t]E_{t-1}[r_{it}]$$

By the definition of correlation, $\rho > -1$. This implies that

$$-1\sigma(m_t)\sigma(r_{it}) \leq E_{t-1}[m_t]E_{t-1}[r_{it}]$$

$$\frac{\sigma(m_t)}{E[m_t]} \geq \frac{E[r_{it}]}{\sigma(r_{it})}$$

Since this will hold for any i, we get that

$$\frac{\sigma(m_t)}{E[m_t]} \geq \max_i \frac{E[r_{it}]}{\sigma(r_{it})}$$

The implication of inequalities like these has been much discussed, the best known is the Hansen and Jagannathan (1991) paper.

9 Time-varying expected returns.

Much work has been expended on showing how we can use known data to predict future returns. In the framework we have discussed, this can be viewed as evidence that the conditional expected return is time-varying, and that old data may be used to generate the conditional expectations.

In the paper, section 3.4 discusses a large number of variables that have been shown to be useful in predicting future returns.

Section 3.5 shows how this is modelled as part of the generation of conditional expectations, called "latent variables."

Section 3.6 discusses a large body of papers, all of which involves modelling explicitly time-variation in other conditional moments than the mean. If the mean $E_t[r_{i,t+1}]$ is assumed to vary over time, we would expect other moments, like the conditional variance $E_t[r_{i,t+1}^2]$ to also vary. This variation over time is modelled using a large number models, examples include ARCH (autoregressive conditional heteroskedasticity), GARCH (generalised ARCH) EGARCH (exponential GARCH) and many others. All of them specifies the current conditional variance as a function of past data.

References

Wayne Ferson. Theory and empirical testing of asset pricing models. In R A Jarrow, V Maksimovic, and W T Ziemba, editors, *Finance*, volume 9 of *Handbooks in Operations Research and Management Science*, chapter 5, pages 145–200. North–Holland, 1995.

Lars Peter Hansen and Ravi Jagannathan. Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99(2):225–62, 1991.

Lars Peter Hansen and Scott F Richard. The role of conditioning information in deducing testable restricions implied by dynamic asset pricing models. *Econometrica*, 55:587–613, 1987.

Robert Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1429–1445, 1978.

David G Luenberger. *Optimization by vector space methods.* Wiley, 1969.