Insight from Merton(1980): The link between period length, observation frequency, and estimator accuracy

Consider: estimation of expected asset returns.
Ask: Can we improve on estimates of expected returns by being clever with the data?
Intuitively: Returns are observed very often,

- daily
- or even more frequent, with todays high frequency datasets)

This may provide *many* observations, which can be used to improve estimates?

Unfortunately, this is *not* the case.

The only thing that improves estimates of mean returns is getting more data in terms of the *total period* observed.
Slicing the returns into more and more short periods will not improve the accuracy of *mean* estimates.
It will however improve accuracy of *variance* estimates.

This important insight is provided in the appendix of Merton (1980).
Let us repeat his argument.
The instantaneous rate of return, dM/M is assumed to follow a diffusion type process.

$$\frac{dM(t)}{M(t)} = \mu dt + \sigma dZ(t) \tag{1}$$

Let μ be the expected return and σ^2 the variance of the return. Suppose these are both constants over a time interval of length I_t, and that the realized return on the market can be observed over time intervals of length Δ where $\Delta << I_t$.

Returns

Let X_k denote the logarithmic return on the market over the k'th observation interval of length Δ during a typical period of length h for $k = 1, 2, \cdots, n$.
X_k kan be written as

$$X_k = \mu\Delta + \sigma\sqrt{\Delta}\varepsilon_k, \quad k = 1, 2, \cdots, n \tag{2}$$

where the $\{\varepsilon_k\}, k = 1, \cdots, n$, are independent and identically distributed standard normal random variables.
Then $n = h/\Delta$ is the number of observations of realized returns over a time interval of length I_t.

Estimating mean

The estimator for the expected logarithmic return

$$\hat{\mu} = \left(\sum_{1}^{n} X_k \right) / h,$$

It has the properties:

$$E[\hat{\mu}] = \mu$$

and

$$\text{var}(\hat{\mu}) = \sigma^2 / h$$

The accuracy of the estimator as measured by $\text{var}(\hat{\mu})$ depend only upon the total length of the observation period h and *not* upon the number of observations n.

That is, nothing is gained in terms of accuracy of the expected return estimate by choosing finer observation intervals for the returns and thereby, increasing the number of observations n for a fixed value of h.

Estimating Variance

Setting the expected return to zero, we estimate variance as:

$$\hat{\sigma}^2 = \frac{1}{h}\sum_{1}^{n} X_k^2$$

Properties of this estimator:

$$E[\hat{\sigma}^2] = \sigma^2 + \mu^2\Delta = \sigma^2 + \mu^2 h/n$$

$$\text{var}(\hat{\sigma}^2) = 2\sigma^4/n + r\mu^2 h/n^2$$

Because the estimator for σ^2 was not taken around the sample mean $\hat{\mu}$, $\hat{\sigma}^2$ is biased.
However, for large n, the difference between the sample second central and non-central moments is trivial.
$\text{var}(\hat{\sigma}^2)$ does depend on the number of observations n for a fixed h, and indeed, to order $1/n$, it depends only upon the number of observations.
By choosing finer observation intervals Δ, the accuracy of the variance estimator can be improved for a fixed value of h."

Robert C Merton. On estimating the expected return on the market. *Journal of Financial Economics*, pages 323–362, 1980.