Lecture Notes based on Merton (1980)

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1 Insight from Merton(1980): The link between period length, observation frequency, and estimator accuracy

We look at estimation of expected asset returns.

Ask: Can we improve on estimates of the expected return by being clever with the data?

Intuitively, one may think that one may use the fact that returns are observed very often, so frequent as daily (or even more frequent, with todays high frequency datasets), may provide *many* observations, which can be used to improve estimates.

Unfortunately, this is not the case.

It turns out that the only thing that improves estimates of mean returns is getting more data in terms of the *total period* we observe. Slicing the returns into more and more short periods will not improve the accuracy of mean estimates.

It will however improve accuracy of *variance* estimates.

This important insight is provided in the appendix of **?**.

Let us repeat his argument.

The instantaneous rate of return, dM/M is assumed to follow a diffusion type process.

$$\frac{dM(t)}{M(t)} = \mu dt + \sigma dZ(t) \tag{1}$$

Let μ be the expected return and σ^2 the variance of the return.

Suppose these are both constants over a time interval of length I_t, and that the realized return on the market can be observed over time intervals of length Δ where $\Delta << I_t$.

Let X_k denote the logarithmic return on the market over the k'th observation interval of length Δ during a typical period of length h for $k = 1, 2, \cdots, n$.

X_k kan be written as

$$X_k = \mu\Delta + \sigma\sqrt{\Delta}\varepsilon_k, \quad k = 1, 2, \cdots, n \tag{2}$$

where the $\{\varepsilon_k\}, k = 1, \cdots, n$, are independent and identically distributed standard normal random variables.

Then $n = h/\Delta$ is the number of observations of realized returns over a time interval of length I_t.

The estimator for the expected logarithmic return $\hat{\mu} = \left(\sum_1^n X_k\right)/h$, will have the properties that

$$E[\hat{\mu}] = \mu$$

and

$$\text{var}(\hat{\mu}) = \sigma^2/h$$

The accuracy of the estimator as measured by $\text{var}(\hat{\mu})$ depend only upon the total length of the observation period h and *not* upon the number of observations n.

That is, nothing is gained in terms of accuracy of the expected return estimate by choosing finer observation intervals for the returns and thereby, increasing the number of observations n for a fixed value of h.

Setting the expected return to zero, we estimate variance as:

$$\hat{\sigma}^2 = \frac{1}{h} \sum_1^n X_k^2$$

This estimator will have the properties that

$$E[\hat{\sigma}^2] = \sigma^2 + \mu^2 \Delta = \sigma^2 + \mu^2 h/n$$

and

$$\text{var}(\hat{\sigma}^2) = 2\sigma^4/n + r\mu^2 h/n^2$$

Because the estimator for σ^2 was not taken around the sample mean $\hat{\mu}$, $\hat{\sigma}^2$ is biased. However, for large n, the difference between the sample second central and non-central moments is trivial.

$\text{var}(\hat{\sigma}^2)$ does depend on the number of observations n for a fixed h, and indeed, to order $1/n$, it depends only upon the number of observations.

By choosing finer observation intervals Δ, the accuracy of the variance estimator can be improved for a fixed value of h."