Finance examples of linear algebra usage

Showing how one can use a matrix handler as a convenient "shorthand" tool for calculations in finance.

Example

(from NHH exam fall 2011)
Consider two one-period economies, $\{P^A, D^A\}$ and $\{P^B, D^B\}$, each with $N = 3$ assets with payoffs across $S = 3$ states. Their prices and payoff matrices $D_{S \times N}$ (each row represents a different state) are given by

$$P^A = \begin{bmatrix} 0.45 \\ 1.75 \\ 1.30 \end{bmatrix}$$

$$P^B = \begin{bmatrix} 0.45 \\ 1.85 \\ 1.60 \end{bmatrix}$$

$$D^A = D^B = D = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

1. What is the present value of the cash flow $Y = (3, 3, 3)'$, and what is the riskless rate of return in the two economies?
2. Are the prices consistent with no arbitrage?

Solution

1. Finding prices

A bit careful here, note that the states are on the rows, so we need to transpose D below:

```
>> D=[0 1 3;0 0 2;1 4 1]
D =
   0   1   3
   0   0   2
   1   4   1
>> Pa=[0.45;1.75;1.3]
Pa =
   0.45000
   1.75000
   1.30000
Pb=[0.45;1.85;1.6]
Pb =
   0.45000
   1.85000
   1.60000
```

We want to find the vector of prices Q that solves

$$P^A = D'Q^A \implies Q^A = (D')^{-1}P^A$$

$$P^B = D'Q^A \implies Q^B = (D')^{-1}P^B$$

```
>> Qa=inv(D')*Pa
Qa =
  -0.050000
   0.500000
   0.450000
>> Qb=inv(D')*Pb
Qb =
   0.050000
   0.500000
   0.450000
```

the risk free rates are found by

$$R_f = \frac{1}{[1\ 1\ 1]Q}$$

```
>> Rfa=1/([1 1 1]*Qa)
Rfa =  1.1111
>> Rfb=1/([1 1 1]*Qb)
Rfb =  1.00000
```

To check that the prices are correct, see whether we get the P's back

```
>> D'*Qa
ans =
   0.45000
   1.75000
   1.30000
>> D'*Qb
ans =
   0.45000
   1.85000
   1.60000
```

Value of cash flow

```
>> [3 3 3]*Qa
ans =  2.7000
>> [3 3 3]*Qb
ans =  3.0000
```

Solution

1. Finding prices

A bit careful here, note that the states are on the rows, so we need to transpose D

Start with economy A:

We want to find the vector of prices Q^A that solves

$$\mathbf{P}^A = \mathbf{D}'\mathbf{Q}^A$$

$$\implies \mathbf{Q}^A = (\mathbf{D}')^{-1}\mathbf{P}^A$$

$$Q^A = \left(\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 1 & 4 & 1 \end{bmatrix}' \right)^{-1} \begin{bmatrix} 0.45 \\ 1.75 \\ 1.30 \end{bmatrix} = \begin{bmatrix} -0.05 \\ 0.5 \\ 0.45 \end{bmatrix}$$

Note that one of the implied state prices is negative, which is inconsistent with no arbitrage.

Solution ctd

Doing the same for economy B

$$\mathbf{Q}^B = (\mathbf{D}')^{-1}\mathbf{P}^B$$

$$\mathbf{Q}^B = \left(\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 1 & 4 & 1 \end{bmatrix}' \right)^{-1} \begin{bmatrix} 0.45 \\ 1.85 \\ 1.60 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.5 \\ 0.45 \end{bmatrix}$$

Solution ctd

The risk free rates are found by

$$R_f = \frac{1}{[1 \;\; 1 \;\; 1]\mathbf{Q}}$$

$$R_f^A = 1/([1 \;\; 1 \;\; 1] * \mathbf{Q}^A) = 1.1111$$

Note that this is a gross rate, so the risk free rate is 11%.

$$R_f^B = 1/([1 \;\; 1 \;\; 1] * \mathbf{Q}^B) = 1,$$

or a interest rate of 0%.

Solution ctd

Value of cash flow:
In economy A:

$$[3 \;\; 3 \;\; 3]\mathbf{Q}^A = 2.7$$

In Economy B:

$$[3 \;\; 3 \;\; 3]\mathbf{Q}^B = 3$$

Solution using a matrix tool

1. Finding state prices

A bit careful here, the states are on the rows, so we need to transpose D:
Defining the inputs

```
>> D=[0 1 3;0 0 2;1 4 1]
D =
   0    1    3
   0    0    2
   1    4    1
```

Solution using a matrix tool ctd

Defining inputs

```
>> Pa=[0.45;1.75;1.3]
Pa =
   0.45000
   1.75000
   1.30000
Pb=[0.45;1.85;1.6]
Pb =
   0.45000
   1.85000
   1.60000
```

Solution using a matrix tool ctd

We want to find the vector of prices Q^A that solves

$$P^A = D'Q^A \implies Q^A = (D')^{-1}P^A$$

```
>> Qa=inv(D')*Pa
Qa =
  -0.050000
   0.500000
   0.450000
```

Solution using a matrix tool ctd

Doing the same for economy B:

$$Q^B = (D')^{-1} P^B$$

```
>> Qb=inv(D')*Pb
Qb =
   0.050000
   0.500000
   0.450000
```

Solution using a matrix tool ctd

the risk free rates are found by

$$R_f = \frac{1}{[1 \;\; 1 \;\; 1]\mathbf{Q}}$$

```
>> Rfa=1/([1 1 1]*Qa)
Rfa =  1.1111
>> Rfb=1/([1 1 1]*Qb)
Rfb =  1.00000
```

Solution using a matrix tool ctd

To check that the prices are correct, see whether we get the **P**'s back

```
>> D'*Qa
ans =
   0.45000
   1.75000
   1.30000
>> D'*Qb
ans =
   0.45000
   1.85000
   1.60000
```

Solution using a matrix tool ctd

Value of cash flow

```
>> [3 3 3]*Qa
ans =  2.7000
>> [3 3 3]*Qb
ans =  3.0000
```

Exercise

There are four possible future states, with state prices

$$\phi = \begin{bmatrix} 0.25 & 0.2 & 0.15 & 0.10 \end{bmatrix}$$

A firm has a set of state contingent cash flows:

$$Cflow = \begin{bmatrix} 100 & 110 & 200 & 500 \end{bmatrix}$$

1. Suppose the firm is all equity financed. Determine the value of the firm and the value of equity.
2. Suppose instead the firm has debt with face value $F = 100$ payable in period 1. Determine the current value of the debt and equity in the company. Is the debt risk free? What is the sum of debt and equity?
3. Suppose instead the firms debt has a face value of $F = 200$. Determine the current value of the debt and equity in the company. Is the debt risk free? What is the sum of debt and equity?

Solution

Let V be the firm's value and E and D the market values of debt and equity.
If all equity: Value of the firms equity is equal to the current value of the cashflows in the different states.

$$V = E = C'\phi = 127$$

```
> C=[100 110 200 500]'
> phi =[ 0.25 0.2 0.15 0.10 ]'
> C'*phi
ans = 127
```

Now consider the case of the firm having debt with face value $F = 100$ payable in period 1.
This debt is risk free, the firm always has cash flows to pay the bond holders the full principal
Cash flows to debt is

$$CF_D \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

$$D = \phi' CF_D = \begin{bmatrix} 0.25 \\ 0.2 \\ 0.15 \\ 0.10 \end{bmatrix}' \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = 70$$

An alternative way to calculate this is to use the risk free rate of interest.

$$R_f = \frac{1}{\sum_\omega \phi_\omega} = \frac{1}{0.25 + 0.2 + 0.15 + 0.1} = \frac{1}{0.7} = 42.86\%$$

Since the debt is riskless, it can be discounted at the risk free rate of interest.

$$D = \frac{100}{1.4286} \approx 70$$

Now, what about equity? The cash flows to equity is what is left over after debt has been paid.

$$CF_E = C - CF_D = \begin{bmatrix} 100 \\ 110 \\ 200 \\ 500 \end{bmatrix} - \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 100 \\ 400 \end{bmatrix}$$

The current value of the firms equity is then

$$E = \phi' CF_E = \phi' \begin{bmatrix} 0 \\ 10 \\ 100 \\ 400 \end{bmatrix} = 57$$

The value of the firm is then

$$V = D + E = 70 + 57 = 127,$$

the same as before.

```
> CF_D = [100;100;100;100]
> D = phi'* CF_D
D = 70
> phi'*ones(1,4)'
ans = 0.70000
> 1.0/0.7
ans = 1.4286
> 100/1.4286
ans = 69.999
> CF_E = C-CF_D
    0
   10
  100
  400
> E=phi'*CF_E
E = 57
> V=D+E
V = 127
```

Consider next a firm with bonds outstanding with face value $F = 200$.

$$CF_D = \min(200, C) = \begin{bmatrix} \min(200, 100) \\ \min(200, 110) \\ \min(200, 200) \\ \min(200, 500) \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 200 \\ 200 \end{bmatrix}$$

$$CF_E = C - CF_D = \begin{bmatrix} 100 - 100 \\ 110 - 110 \\ 200 - 200 \\ 500 - 200 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 300 \end{bmatrix}$$

The values of debt and equity are

$$D(F = 200) = \phi' CF_D = 97$$

$$E(F = 200) = \phi' CF_E = 30$$

$$V(F = 200) = D + E = 97 + 30 = 127$$

No matter what the mix of debt and equity, the value of the firm stays at 127.

```
> CF_D = min(200,C)
  100
  110
  200
  200
> CF_E = C-CF_D
    0
    0
    0
  300
> D=phi'*CF_D
D = 97
> E=phi'*CF_E
E = 30
> V=D+E
V = 127
```

Solution

1. Suppose the firm is all equity financed. Determine the value of the firm and the value of equity.

Let V be the firm's value and E and D the market values of debt and equity.
If all equity: Value of the firms equity is equal to the current value of the cashflows in the different states.

$$V = E = C'\phi$$

$$= \begin{bmatrix} 100 \\ 110 \\ 200 \\ 500 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.2 \\ 0.15 \\ 0.10 \end{bmatrix} = 127$$

Solution ctd

2. Suppose instead the firm has debt with face value $F = 100$ payable in period 1. Determine the current value of the debt and equity in the company. Is the debt risk free? What is the sum of debt and equity?

Start with valuing debt.

If the face value $F = 100$, this debt is risk free, the firm always has cash flows to pay the bond holders the full principal

Cash flows to debt is

$$CF_D = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

Value of debt

$$D = \phi' CF_D = \begin{bmatrix} 0.25 \\ 0.2 \\ 0.15 \\ 0.10 \end{bmatrix}' \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = 70$$

Solution ctd

An alternative way to calculate this is to use the risk free rate of interest.

$$R_f = \frac{1}{\sum_\omega \phi_\omega} = \frac{1}{0.25 + 0.2 + 0.15 + 0.1} = \frac{1}{0.7} = 42.86\%$$

Since the debt is riskless, it can be discounted at the risk free rate of interest.

$$D = \frac{100}{1.4286} \approx 70$$

Solution ctd

Now, what about equity? The cash flows to equity is what is left over after debt has been paid.

$$CF_E = C - CF_D = \begin{bmatrix} 100 \\ 110 \\ 200 \\ 500 \end{bmatrix} - \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 100 \\ 400 \end{bmatrix}$$

The current value of the firms equity is then

$$E = \phi' CF_E = \phi' \begin{bmatrix} 0 \\ 10 \\ 100 \\ 400 \end{bmatrix} = 57$$

The value of the firm is then

$$V = D + E = 70 + 57 = 127,$$

the same as before.

Solution ctd

Consider next a firm with bonds outstanding with face value $F = 200$.

$$CF_D = \min(200, C) = \begin{bmatrix} \min(200, 100) \\ \min(200, 110) \\ \min(200, 200) \\ \min(200, 500) \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 200 \\ 200 \end{bmatrix}$$

$$CF_E = C - CF_D = \begin{bmatrix} 100 - 100 \\ 110 - 110 \\ 200 - 200 \\ 500 - 200 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 300 \end{bmatrix}$$

The values of debt and equity are

$$D(F = 200) = \phi' CF_D = 97$$

$$E(F = 200) = \phi' CF_E = 30$$

$$V(F = 200) = D + E = 97 + 30 = 127$$

No matter what the mix of debt and equity, the value of the firm stays at 127

Solving using a matrix tool

1. Suppose the firm is all equity financed. Determine the value of the firm and the value of equity.

```
> C=[100 110 200 500]'
> phi =[ 0.25 0.2 0.15 0.10 ]'
> C'*phi
ans = 127
```

Solving using a matrix tool ctd

2. Suppose instead the firm has debt with face value $F = 100$ payable in period 1. Determine the current value of the debt and equity in the company. Is the debt risk free? What is the sum of debt and equity?

```
> CF_D = [100;100;100;100]
> D = phi'* CF_D
D = 70
```

Solving using a matrix tool ctd

Valuing debt using the risk free interest rate

```
> phi'*ones(1,4)'
ans = 0.70000
> 1.0/0.7
ans = 1.4286
> 100/1.4286
ans = 69.999
```

Solving using a matrix tool ctd

Valuing equity when the principal of debt is 100.

```
> CF_E = C-CF_D
    0
   10
  100
  400
> E=phi'*CF_E
E = 57
> V=D+E
V = 127
```

Solving using a matrix tool ctd

3. Suppose instead the firms debt has a face value of $F = 200$. Determine the current value of the debt and equity in the company. Is the debt risk free? What is the sum of debt and equity?

First, finding the cash flows of debt and valuing it.

```
> CF_D = min(200,C)
  100
  110
  200
  200
```

Solving using a matrix tool ctd

Then finding equity:

```
> CF_E = C-CF_D
    0
    0
    0
  300
> D=phi'*CF_D
D = 97
> E=phi'*CF_E
E = 30
> V=D+E
V = 127
```

Example

You observe the three risk free bonds A, B and C:

Bond	Price	Cashflow in period 1	2	3
A	95	100	0	0
B	90	10	110	0
C	85	10	10	110

1. What is the current value of receiving one dollar at time 3?
2. What are the interest rates (with annual compounding) implied in these prices?

Another risk free bond D is traded, with the following cash flows:

time:	1	2	3
D	20	20	520

2. What is the current price of bond D?

Solution

1. One way to think of this is using state prices. Want to calculate the price of receiving one dollar for certain in period (state) t.

If

```
B =
   100     0     0
    10   110     0
    10    10   110
```

is the payoffs, and

```
P =
   95
   90
   85
```

is the current prices of the bonds,

find the current state prices d (discount factors) from the relation:

$$P = Bd$$

$$d = B^{-1}P$$

```
d = inv(B)*P
  0.95000
  0.73182
  0.61983
```

The current value of receiving one dollar at time 3 is thus 0.61983.

2. To find the interest rates we solve the expressions

$$d_t = \frac{1}{(1+r)^t}$$

wrt r. Doing everything at once:

```
>> t=[1;2;3];
>> r=d.^(-1./t)-1
r =
  0.052632
  0.168957
  0.172847
```

The interest rates are increasing with time. To check that these are correct, recalculate

```
>> (1./(1+r)).^t
ans =
  0.95000
  0.73182
  0.61983
```

To price bond D we use the “discount factors” d:

```
> D =
   20    20   520
> d =
  0.95000  0.73182  0.61983
D*d' = 355.95
```

Bond D has a price equal to 355.95.

Solution

1. One way to think of this is using state prices. Want to calculate the price of receiving one dollar for certain in period (state) t.

Define the matrices

$$\mathbf{B} = \begin{bmatrix} 100 & 0 & 0 \\ 10 & 110 & 0 \\ 10 & 10 & 110 \end{bmatrix}$$

as the payoffs, and

$$\mathbf{P} = \begin{bmatrix} 95 \\ 90 \\ 85 \end{bmatrix},$$

the current prices of the bonds.

Solution ctd

Find the current state prices $\mathbf{d}$ (discount factors) from the relation:

$$\mathbf{P} = \mathbf{B}\mathbf{d}$$

$$\mathbf{d} = \mathbf{B}^{-1}\mathbf{P}$$

$$\mathbf{d} = \begin{bmatrix} 100 & 0 & 0 \\ 10 & 110 & 0 \\ 10 & 10 & 110 \end{bmatrix}^{-1} \begin{bmatrix} 95 \\ 90 \\ 85 \end{bmatrix} = \begin{bmatrix} 0.95000 \\ 0.73182 \\ 0.61983 \end{bmatrix}$$

The current value of receiving one dollar at time 3 is 0.61983.

Solution ctd

2. What are the interest rates (with annual compounding) implied in these prices?

To find the interest rates we solve the expressions

$$d_t = \frac{1}{(1 + r_t)^t}$$

wrt r_t.

$$\mathbf{r} = \begin{bmatrix} 0.052632 \\ 0.168957 \\ 0.172847 \end{bmatrix}$$

Solution ctd

3. What is the current price of bond D?

Define the payoff matrix for bond D as

$$\mathbf{D} = \begin{bmatrix} 20 \\ 20 \\ 520 \end{bmatrix}$$

To price bond D we use the "discount factors" **d**:

$$\text{Price D} = \begin{bmatrix} 20 \\ 20 \\ 520 \end{bmatrix}' \begin{bmatrix} 0.95000 \\ 0.73182 \\ 0.61983 \end{bmatrix} = 355.95$$

Bond D has a price equal to 355.95.

Solving using a matrix tool

1. Discount factors

Defining the inputs

```
B =
   100     0     0
    10   110     0
    10    10   110
```

is the payoffs, and

```
P =
   95
   90
   85
```

is the current prices of the bonds,

Solving using a matrix tool ctd

Find the current state prices d (discount factors)

```
d = inv(B)*P
  0.95000
  0.73182
  0.61983
```

Solving using a matrix tool ctd

2. To find the interest rates we solve the expressions

$$d_t = \frac{1}{(1 + r_t)^t}$$

wrt r:

```
>> t=[1;2;3];
>> r=d.^(-1./t)-1
r =
  0.052632
  0.168957
  0.172847
```

Solving using a matrix tool ctd

To check that these are correct, recalculate the discount factors:

```
>> (1./(1+r)).^t
ans =
  0.95000
  0.73182
  0.61983
```

Solving using a matrix tool ctd

3. Price bond B

Define bond payoffs

```
> D =
   20    20   520
```

To price bond D we use the "discount factors" *d* already calculated

```
> d =
  0.95000  0.73182  0.61983
D*d' = 355.95
```

Bond D has a price equal to 355.95.