We will consider the testing of the rational expectations asset pricing model of Lucas (1978), using GMM as a testing device, which is the application done by Hansen and Singleton (1982). We will consider a simplified version of this application.

We posit the existence of a *representative consumer* who is maximising his (or hers) expected utility of future consumption. Let $c_t$ be the consumption in period $t$. There is only one asset in the economy, with price $p_t$ and paying dividends of $d_t$ in period $t$. Let $q_t$ be the agents holdings (quantity) of the asset at the beginning of period $t$. The consumer is assumed to have wage income of $w_t$.

1. Verify that the agents *budget constraint* is

$$c_t + p_t q_t \leq (p_t + d_t) q_{t-1} + w_t$$

The consumer is assumed to maximise his lifetime expected utility

$$E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right]$$

where $\beta$ is a discount factor. We assume is that the amount of productive asset (the tree) is fixed. We can thus close this model by noting that in equilibrium, the demand of assets is equal to the supply, and since we have only one agent, $q_t = q_{t+1}$ for all $t$. The problem we want to solve is then

$$\max_{\{c_t, q_t\}} E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right] \text{ subject to } c_t + p_t q_t \leq (p_t + d_t) q_{t-1} + w_t \text{ for}$$

2. Show that the maximization problem faced by the consumers is represented by the following Lagrangian

$$L = E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right] - E_0 \left[ \sum_{t=1}^{T} \lambda_t \left( c_t + p_t q_t - (p_t + d_t) q_{t-1} - w_t \right) \right]$$

Note that we here write $E_t[\cdot]$, which is shorthand for the expected value, conditional on the information set at time t, which could also be written as $E[\cdot|I_t]$, where $I_t$ is the *information set*, the information available to the decision maker at time $t$. The fact that we are using conditional expectations is important, as you will see shortly.

3. By manipulating the first order conditions of the Lagrangian, show that the solution to the problem satisfies:

$$E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\frac{(p_{t+1}+d_{t+1})}{p_t}\right]=1$$

Showing the first order conditions: Taking derivatives wrt $c_r$ and $q_r$ we get

$$\begin{aligned}
\frac{\partial L}{\partial c_r} &= E_0\left[\beta^r u'(c_r)\right] - \lambda_r &= 0 \\
\frac{\partial L}{\partial q_r} &= -\lambda_r p_r + \lambda_{r+1}(p_{r+1} + d_{r+1}) &= 0
\end{aligned}$$

Use the first equation to substitute in the second, and we get a condition for optimality that will need to hold for any $c_t$.

$$E_t[\beta^t u'(c_t) p_t] = E_t\left[\beta^{t+1} u'(c_{t+1})(d_{t+1} + p_{t+1})\right]$$

or

$$E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{(p_{t+1} + d_{t+1})}{p_t}\right] = 1$$

This is usually called the *Euler equation* in this type of model. The economic interpretation of this model is that $\frac{u'(c_{t+1})}{u'(c_t)}$ is the marginal rate of substitution $mrs(c_t, c_{t+1})$ for consumption from period $t$ to period $t+1$, and $\frac{(p_{t+1}+d_{t+1})}{p_t}$ the return $R_t$ on the asset in period $t$, i.e. the testable implication is that

$$E\left[mrs\left(c_t, c_{t+1}\right) R_{t+1}\right] = 1$$

This equality will hold for all times $t$.

Let us now discuss estimation in this context. Suppose we want to apply this to aggregate data for a real economy. What if we use aggregate stock returns and dividends to proxy for the asset, and use consumption per capita as consumption data. These are all available time series. So if we look at the equation

$$E\left[mrs\left(c_t, c_{t+1}\right) R_t\right] = 1$$

The problem we find is that we do not know the functional form of the *mrs*, (i.e. the utility function $u(c_t)$, and the equality only hold in expectation.

To do something about the first problem we have to parameterize the utility function by some functional form. Suppose the utility function is a *power* utility function of the form:

$$u(c_t) = \frac{1}{1-\alpha}\left(c^{\alpha} - 1\right)$$

4. Show that the marginal rate of substitution has the form

$$mrs(c_t, c_{t+1}) = \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1}$$

Calculating the *mrs*:

$$u'(c_t) = \frac{\alpha}{1-\alpha} c^{\alpha-1}$$

giving

$$mrs(c_t, c_{t+1}) = \frac{u'(c_{t+1})}{u'(c_t)} = \frac{\frac{\alpha}{1-\alpha} c_{t+1}^{\alpha-1}}{\frac{\alpha}{1-\alpha} c_t^{\alpha-1}}$$

$$= \frac{c_{t+1}^{\alpha-1}}{c_t^{\alpha-1}} = \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1}$$

The testable implication of the model is then that

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha - 1} \frac{(p_{t+1} + d_{t+1})}{p_t} \right] = 1$$

or

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha - 1} \frac{(p_{t+1} + d_{t+1})}{p_t} - 1 \right] = 0$$

that is, the conditional expectation of the above equals zero. We are interested in estimating the two parameters $\alpha$ and $\beta$, and a nonlinear relation in these two variables, and an equation that has expectation zero under the true parameters, (say) $\alpha_0$ and $\beta_0$.

Consider now some variables $Z_t$ that are in the information set at time $t$ (known at time $t$.) We impose rational expectations, which in this case means that all information is being used to form optimal conditional expectations.
The variables $Z_t$, since they are known at time $t$, can be viewed as constants relative to the conditional expectation, and we can "multiply through" the expectation

$$E_t\left[\left\{\beta\left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1}\frac{(p_{t+1}+d_{t+1})}{p_t}-1\right\}Z_t\right]=\mathbf{0}$$

Intuitively, any variables $Z_t$ in the information set at time $t$ have been used to generate the expectation, so they should not be able to explain anything more, and should thus be orthogonal.

What we now have is a situation where we have two unknown parameters, $\beta$ and $\alpha$. $\beta$ is the *discount factor*, and $\alpha$ the *risk aversion* parameter, and we have an expression which has expectation equal to zero for each period. Intuitively, the obvious thing to do is to replace the expectation with the sample average, and find the parameters $\alpha$ and $\beta$ that sets the sample average to zero. In order for this to make sense, we need to make assumptions that ensures that the sample average will going to converge to the expectation,

$$\frac{1}{T}\sum_{t=1}^{T} x_t \to E[x]$$

using some mode of convergence.

The equation gives "moment restrictions" with conditional expectation zero. As "instruments" $Z_t$ we can pick any variables $z_t$ in the information set at time $t$. If we pick two variables, the parameters are exactly identified. As examples, pick the two instruments

1. Unity. $(Z_{1t} = 1)$
2. Last periods returns. $(Z_{2t} = r_{t-1})$.

Then the equation to be tested is:

$$E_t\left[\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1}\frac{p_{t+1}+d_{t+1}}{p_t}-1\right]\left[\begin{array}{c}1\\ r_{t-1}\end{array}\right]\right]=\mathbf{0}$$

# Exercise

The solution to a representative agent asset pricing model reduces to

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha - 1} \frac{p_{t+1} + d_{t+1}}{p_t} - 1 \right] = \mathbf{0}$$

where $E_t[]$ signfifies the conditional expectation at date $t$, $c$ consumption (per capita), $p$ stock prices and $d$ dividends.

# Exercise

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# Exercise

Then we can use the following set of moment conditions for estimation of an exactly identified system

$$E_t \left[ \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha-1} \frac{p_{t+1} + d_{t+1}}{p_t} - 1 \right] \left[ \begin{array}{c} 1 \\ r_{t-1} \end{array} \right] \right] = \mathbf{0}$$

This equation has been thoroughly studied, from Hansen and Singleton (1982) onwards, particularly for the US.

# Exercise

We want to do a similar exercise for Norway. To that end we need data for per capital consumption and stock returns. TThe highest frequency we can get consumption data is quarterly. Estimate per capita consumption and quarterly stock returns, test this model using GMM on data for Norway.
You will need to get data for consumption and population from the bureau of statistics. The stock return is the quarterly return on some broad stock market index.

# Exercise Solution

Reading in the data and aligning it in time is a bit of a chore.

```
Consum <- read.table("../../../../data/norway/economic_statistics/konsum.csv",h
cons <- ts(Consum[,2],frequency=4,start=c(1978,1));
Pop  <- read.table("../../../../data/norway/economic_statistics/folkemengde_qua
pop <- ts(Pop[,2],frequency=4,start=c(1978,1))
pc <- cons/pop
Ret  <- read.table("../../../../data/norway/stock_market_indices/market_portfol
ew <-  ts(Ret[,2],frequency=4,start=c(1980,3));
ewlag <- lag(ew,1);
RelC <- pc/lag(pc,1)
Int = ts.intersect(RelC,ew,ewlag)
X = as.vector(Int[,1]);
X = cbind(X,as.vector(Int[,2]))
X = cbind(X,as.vector(Int[,3]))
dim(X)
```

But we end up with a X matrix with 3 aligned vectors: Consumption growth, stock returns, and one period lagged stock returns. This is used in the function specifying the moment condition.

```
# moment conditions, FOC for consumers problem
# parameter beta, alpha, instruments: unity, lagged return
g <- function (parms,x) {
  beta <- parms[1];
  alpha <- parms[2];
  m1 <- beta * X[,1] ^ (alpha -1) * (1+X[,2])-1
  m2 <- m1*X[,3];
  f <- cbind(m1,m2)
  return (f);
}
```

Once the moment conditions are specified running GMM is merely a matter of

```
library(gmm)
t0=c(1,5);
res=gmm(g,X,t0)
summary(res)
```

Which produces the output

```
> dim(X)
[1] 122   3
```

The estimation uses data starting in june 1980 with 122 quarterly observations.

```
gmm(g = g, x = X, t0 = t0)
Method:  twoStep
Kernel:  Quadratic Spectral
Coefficients:
          Estimate     Std. Error   t value      Pr(>|t|)
Theta[1]   1.0009e+00   2.7755e-02   3.6060e+01  9.6058e-28
Theta[2]   7.1276e+00   1.2182e+01   5.8507e-01   5.5850e-0
J-Test: degrees of freedom is 0
                J-test                 P-value
Test E(g)=0:    4.44397617550074e-11  *******
```

The output of the GMM estimation is the parameter estimates

$$\alpha = 7.12$$

$$\beta = 1.0009$$

These are typical values in such an estimation. But the $\beta$ by any economic reasoning needs to be below one (discounting).
Note the reported test for overidentifying restrictions.

Lars P Hansen and Kenneth Singleton. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica*, 50: 1269–1286, September 1982.