

# The Hansen Singleton analysis

November 24, 2021

## 1 Estimation of macroeconomic rational expectations model.

The Hansen Singleton (1982) paper.

We start by looking at the application of GMM that first showed how useful this particular strategy in estimation is. This is the paper of Hansen and Singleton (1982). I will go through a somewhat simplified version of the model in that paper.

The model is a rational expectations, general equilibrium model, first shown in Lucas (1978). We posit the existence of a *representative consumer* who is maximising his (or hers) expected utility of future consumption.

Let  $c_t$  be the consumption in period  $t$ . There is only one asset in the economy,<sup>1</sup> with price  $p_t$  and paying dividends of  $d_t$  in period  $t$ . Let  $q_t$  be the agents holdings (quantity) of the asset at the beginning of period  $t$ . The consumer is assumed to have wage income of  $w_t$ .

It should be easy to verify that the agents *budget constraint* is

$$c_t + p_t q_t \leq (p_t + d_t)q_{t-1} + w_t$$

The consumer is assumed to maximise his lifetime expected utility

$$E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right]$$

where  $\beta$  is a discount factor.

The assumption of the model is that the amount of productive asset (the tree) is fixed. We can thus close this model by noting that in equilibrium, the demand of assets is equal to the supply, and since we have only one agent,  $q_t = q_{t+1} \forall t$ .

The problem we want to solve is then

$$\max_{\{c_t, q_t\}} E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$c_t + p_t q_t \leq (p_t + d_t)q_{t-1} + w_t$$

for  $t = 0, 1, 2, \dots$ .

This problem can be solved in a number of ways, the most standard being by dynamic programming. But let us look at what may be the simplest, doing the optimization directly by forming a Lagrangian:<sup>2</sup>

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<sup>1</sup>This asset is often called a *tree*, because we think of the tree as the only productive asset, and apples as the only product. There is of course historical evidence for the feasibility of this particular “tree” economy.

<sup>2</sup>In this we are not using the usual dynamic formulation. We need some additional restrictions on the solution for this Lagrangian approach to be correct. We call these restrictions transversality conditions. In this example, let us assume the transversality conditions hold, this is not a macro class.

$$L = E_0 \left[ \sum_{t=1}^{\infty} \beta^t u(c_t) \right] - E_0 \left[ \sum_{t=1}^T \lambda_t (c_t + p_t q_t - (p_t + d_t) q_{t-1} - w_t) \right]$$

Note how we here write  $E_t[\cdot]$ , which is shorthand for the expected value, conditional on the information set at time  $t$ , which could also be written as  $E[\cdot|I_t]$ , where  $I_t$  is the *information set*, the information available to the decision maker at time  $t$ . The fact that we are using conditional expectations is important, as you will see shortly.

Take derivatives wrt  $c_r$  and  $q_r$  we get

$$\begin{aligned} \frac{\partial L}{\partial c_r} &= E_0 [\beta^r u'(c_r)] - \lambda_r = 0 \\ \frac{\partial L}{\partial q_r} &= -\lambda_r p_r + \lambda_{r+1} (p_{r+1} + d_{r+1}) = 0 \end{aligned}$$

Use the first equation to substitute in the second, and we get a condition for optimality that will need to hold for any  $c_t$ .

$$E_t[\beta^t u'(c_t) p_t] = E_t [\beta^{t+1} u'(c_{t+1}) (d_{t+1} + p_{t+1})]$$

or

$$E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{(p_{t+1} + d_{t+1})}{p_t} \right] = 1 \tag{1}$$

This is usually called the *Euler equation* in this type of model.

The economic interpretation of this model is that

$$\frac{u'(c_{t+1})}{u'(c_t)} = mrs(c_t, c_{t+1})$$

is the marginal rate of substitution for consumption from period  $t$  to period  $t + 1$ , and

$$\frac{(p_{t+1} + d_{t+1})}{p_t} = R_t$$

is the *return* on the asset in period  $t$ .

The equality 1 will hold for all times  $t$ . Let us now discuss estimation in this context. Suppose we want to apply this to aggregate data for a real economy. What if we use stock returns and dividends to proxy for the asset, and use consumption per capita as consumption data. These are all available time series. So if we look at the equation

$$E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{(p_{t+1} + d_{t+1})}{p_t} \right] = 1$$

The only problem we find is that we do not know the functional form of the utility function  $u(c_t)$ , and the equality only hold in expectation.

To do something about the first problem we have to parameterize the utility function by some functional form. Let us for example use a *power* utility function of the form.

$$u(c_t) = \frac{1}{1 - \alpha} (c_t^\alpha - 1)$$

Then

$$u'(c_t) = \frac{\alpha}{1-\alpha} c_t^{\alpha-1}$$

and

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\frac{\alpha}{1-\alpha} c_{t+1}^{\alpha-1}}{\frac{\alpha}{1-\alpha} c_t^{\alpha-1}} = \frac{c_{t+1}^{\alpha-1}}{c_t^{\alpha-1}} = \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1}$$

The testable implication of the model is then that

$$E_t \left[ \beta \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1} \frac{(p_{t+1} + d_{t+1})}{p_t} \right] = 1$$

or

$$E_t \left[ \beta \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1} \frac{(p_{t+1} + d_{t+1})}{p_t} - 1 \right] = 0$$

that is, the conditional expectation of the above equals zero. We are interested in estimating the two parameters  $\alpha$  and  $\beta$ , and a nonlinear relation in these two variables, and an equation that has expectation zero under the true parameters, (say)  $\alpha_0$  and  $\beta_0$ .

Consider now some variables  $Z_t$  that are in the information set at time  $t$  (known at time  $t$ .) We impose rational expectations, which in this case means that all information is being used to form optimal conditional expectations.

The variables  $Z_t$ , since they are known at time  $t$ , can be viewed as constants relative to the conditional expectation, and we can “multiply through” the expectation:<sup>3</sup>

$$E_t \left[ \left\{ \beta \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1} \frac{(p_{t+1} + d_{t+1})}{p_t} - 1 \right\} Z_t \right] = \mathbf{0}$$

Intuitively, any variables  $Z_t$  in the information set at time  $t$  have been used to generate the expectation, so they should not be able to explain anything more, and should thus be orthogonal.

What we now have is a situation where we have two unknown parameters,  $\beta$  and  $\alpha$ .  $\beta$  is the *discount factor*, and  $\alpha$  the *risk aversion* parameter, and we have an expression which has expectation equal to zero for each period. Intuitively, the obvious thing to do is to replace the expectation with the sample average, and find the parameters  $\alpha$  and  $\beta$  that sets the sample average to zero. In order for this to make sense, we need to make assumptions that ensures that the sample average will going to converge to the expectation,

$$\frac{1}{T} \sum_{t=1}^T x_t \rightarrow E[x]$$

using some mode of convergence.

The equation gives “moment restrictions” with conditional expectation zero. As “instruments”  $Z_t$  we can pick any variables  $z_t$  in the information set at time  $t$ . If we pick two variables, the parameters are exactly identified. As examples, pick the two instruments

1. Unity. ( $Z_{1t} = 1$ )
2. Last periods returns. ( $Z_{2t} = r_{t-1}$ ).

Then

$$E_t \left[ \left[ \beta \left(\frac{c_{t+1}}{c_t}\right)^{\alpha-1} \frac{p_{t+1} + d_{t+1}}{p_t} - 1 \right] \begin{bmatrix} 1 \\ r_{t-1} \end{bmatrix} \right] = \mathbf{0}$$

This equation has been thoroughly studied, from Hansen and Singleton (1982) onwards.

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<sup>3</sup>This is again using the rules about conditional expectations. In this case if  $X$  is random, and  $z \in I_t$ , then  $zE[X|I_t] = E[zX|I_t]$

## 1.1 Results.

Need data:

- Consumption data for an “average consumer.” Find this from the total consumption data, divided the number of people in the economy, from aggregate statistics.
- Stock returns for several indices.

Period: 1953–78.

Find economically plausible estimates of  $\alpha$  and  $\beta$ .<sup>4</sup> However, tend to reject the over-identifying restrictions.

## 1.2 Asset pricing: Consumption based model

### Exercise 1.

The solution to a representative agent asset pricing model reduces to

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha-1} \frac{p_{t+1} + d_{t+1}}{p_t} - 1 \right] = \mathbf{0}$$

where  $E_t[\cdot]$  signifies the conditional expectation at date  $t$ ,  $c$  consumption (per capita),  $p$  stock prices and  $d$  dividends.

The equation gives “moment restrictions” with conditional expectation zero. As “instruments”  $Z_t$  one can pick any variables  $z_t$  in the information set at time  $t$ . If we pick two variables, the parameters are exactly identified. As examples, pick the two instruments

1. Unity. ( $Z_{1t} = 1$ )
2. Last periods returns. ( $Z_{2t} = r_{t-1}$ ).

Then we can use the following set of moment conditions for estimation of an exactly identified system

$$E_t \left[ \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha-1} \frac{p_{t+1} + d_{t+1}}{p_t} - 1 \right] \begin{bmatrix} 1 \\ r_{t-1} \end{bmatrix} \right] = \mathbf{0}$$

This equation has been thoroughly studied, from Hansen and Singleton (1982) onwards, particularly for the US.

We want to do a similar exercise for Norway. To that end we need data for per capital consumption and stock returns. The highest frequency we can get consumption data is quarterly. Estimate per capita consumption and quarterly stock returns, test this model using GMM on data for Norway.

You will need to get data for consumption and population from the bureau of statistics. The stock return is the quarterly return on some broad stock market index.

### Solution to Exercise 1.

Reading in the data and aligning it in time is a bit of a chore.

```
Consum <- read.table(".././.././data/norway/economic_statistics/konsum.csv",header=TRUE,sep="");
cons <- ts(Consum[,2],frequency=4,start=c(1978,1));
Pop <- read.table(".././.././data/norway/economic_statistics/folkemengde_quarterly.csv",header=TRUE,sep="");
pop <- ts(Pop[,2],frequency=4,start=c(1978,1))
pc <- cons/pop
Ret <- read.table(".././.././data/norway/stock_market_indices/market_portfolio_returns_quarterly.txt",header=TRUE);
ew <- ts(Ret[,2],frequency=4,start=c(1980,3));
ewlag <- lag(ew,1);
RelC <- pc/lag(pc,1)
```

<sup>4</sup>If you ever try to replicate these numbers, find the erratum in Econometrica a couple of years later.

```

Int = ts.intersect(ReIc,ew,ewlag)
X = as.vector(Int[,1]);
X = cbind(X,as.vector(Int[,2]))
X = cbind(X,as.vector(Int[,3]))
dim(X)

```

But we end up with a X matrix with 3 aligned vectors: Consumption growth, stock returns, and one period lagged stock returns. This is used in the function specifying the moment condition.

```

# moment conditions, FOC for consumers problem
# parameter beta, alpha, instruments: unity, lagged return
g <- function (parms,x) {
  beta <- parms[1];
  alpha <- parms[2];
  m1 <- beta * X[,1] ^ (alpha -1) * (1+X[,2])-1
  m2 <- m1*X[,3];
  f <- cbind(m1,m2)
  return (f);
}

```

Once the moment conditions are specified running GMM is merely a matter of

```

library(gmm)
t0=c(1,5);
res=gmm(g,X,t0)
summary(res)

```

Which produces the output

```

> dim(X)
[1] 122  3

```

The estimation uses data starting in june 1980 with 122 quarterly observations.

```

gmm(g = g, x = X, t0 = t0)
Method: twoStep
Kernel: Quadratic Spectral
Coefficients:
      Estimate      Std. Error    t value    Pr(>|t|)
Theta[1]  1.0009e+00  2.7755e-02  3.6060e+01  9.6058e-285
Theta[2]  7.1276e+00  1.2182e+01  5.8507e-01  5.5850e-01
J-Test: degrees of freedom is 0
      J-test      P-value
Test E(g)=0:  4.44397617550074e-11  *****

```

The output of the GMM estimation is the parameter estimates

$$\alpha = 7.12$$

$$\beta = 1.0009$$

These are typical values in such an estimation. But the  $\beta$  by any economic reasoning needs to be below one (discounting).

Note the reported test for overidentifying restrictions.

## References

Lars P Hansen and Kenneth Singleton. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica*, 50:1269–1286, September 1982.

Robert Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1429–1445, 1978.