Multivariate Tests of the CAPM

When we for example use the Black Jensen Approach, testing for $\alpha_i = 0$ in

$$r_{it} - r_{ft} = \alpha_i + (r_{mt} - r_{ft}) + \varepsilon$$

on an equation by equation basis, this is inefficient.
Want to aggregate the tests used in e.g. Black et al. [1972] into a single test statistic. If we are willing to make distributional assumptions, in this case multivariate normality, can use Maximum Likelihood methods to construct an aggregate test.
This was developed in a sequence of papers: Gibbons [1982], MacKinlay [1987] and Gibbons et al. [1989].
We talk about the first [Gibbons, 1982] and last [Gibbons et al., 1989] of these papers.

Multivariate tests in a normal setting.

We use the Gibbons paper to show how to write the problem in matrix form, and construct a single system for testing.
Define

$$\tilde{R}_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ \vdots \\ R_{iT} \end{bmatrix}, \quad i_T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{R}_m = \begin{bmatrix} R_{m1} \\ R_{m2} \\ \vdots \\ R_{mT} \end{bmatrix}, \quad \text{and } \tilde{\eta}_i = \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{iT} \end{bmatrix}$$

η_i is the error term, and in the paper this is assumed independently, normally distributed.

$$\tilde{\eta}_i \sim \mathcal{N}(0, \sigma_{ii} I_T)$$

and we are looking at

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

This is the same setup as in Black et al. [1972].
The CAPM imposes

$$\tilde{R}_i = \tilde{r}_{zc} + \beta_i(\tilde{R}_m - \tilde{r}_{zc}) = \tilde{r}_{zc}(1 - \beta_i) + \beta_i \tilde{R}_m$$

If the CAPM is true

$$\tilde{R}_i = \tilde{r}_{zc}(1 - \beta_i) + \beta_i \tilde{R}_m$$

holds for all securities.
If we estimate

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

to test the CAPM, we test whether

$$\mathcal{H}_0 : \alpha_i = r_{zc}(1 - \beta_i) \ \forall \ i$$

against

$$\mathcal{H}_A : \alpha_i \neq r_{zc}(1 - \beta_i) \ \forall \ i$$

The problem is that we do not know r_{zc}, it must be estimated from the data. But then the estimation should take acount of that, under the null, r_{zc} is the same across securities. It is therefore helpful to stack the whole estimation into *one* set of equations.

We can stack the matrices in the following manner

$$\begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix} = \begin{bmatrix} \tilde{(}i_T : R_m) & 0 & \cdots & 0 \\ \tilde{0} & (i_T : R_m) & & \\ \vdots & & \ddots & \\ \tilde{0} & & & (i_T : R_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}$$

and

$$(i_T : R_m) = \begin{bmatrix} 1 & R_{m1} \\ 1 & R_{m2} \\ \vdots & \\ 1 & R_{mT} \end{bmatrix}$$

The system can be even more compactly written using Kroenecker products
as

$$\tilde{R}^* = [(i_T : R_m) \otimes I_N] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \tilde{\eta}^*$$

where we have defined

$$\eta^* = \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix} \quad and \quad \tilde{R}^* = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix}$$

$(1 \times T \cdot N)$ vectors

Kroenecker product $\otimes$:

$$A = \begin{pmatrix} a_{11} & a_{21} & & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ & & \ddots & \\ a_{1n} & & & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{21} & & b_{p1} \\ b_{12} & b_{22} & & b_{p2} \\ & & \ddots & \\ b_{1q} & & & b_{pq} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{21}B & & a_{m1}B \\ a_{12}B & a_{22}B & & a_{m2}B \\ & & \ddots & \\ a_{1n}B & & & a_{mn}B \end{pmatrix}$$

A is a $mp \times nq$ matrix

The null hypothesis involves N variables $\alpha_1, \cdots, \alpha_N$.
Using the classical test statistics:
Wald: Estimate all of the α_i, β_i's. Then test

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LM: Estimate one α_i, say α^*. Then test relaxation of

$$\alpha^* = \alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LR: Use both restricted and urestricted estimates, compare fit.

Multivariate test of the CAPM - GRS

Gibbons et al. [1989] uses the setup of Gibbons [1982] to construct a test statistic to answer only one question, whether the market portfolio m mean variance efficient.

How to test for aggregate MV efficiency:

Consider the estimation of the two following models:

Unconstrained model

$$r_{jt} = \alpha_j + \beta_j r_{mt} + e_{jt}$$

Constrained model

$$r_{jt} = r_{zt}(1 - \beta_j) + \beta_j r_{mt} + e_{jt}$$

The constrained model is a special case of the unconstrained model.

If the CAPM is true, and m *is* MV efficient, the constrained model is the true model. Hence, our estimate of α_j in the unconstrained model should be approximately equal to $r_{zt}(1 - \beta_j)$ (the intercept in the constrained model)

The multivariate tests of MV efficiency compares the fit of these two models.

If the difference is large (according to some statistical metric), reject MV efficiency. Otherwise accept it.

These test statistics relies on using Maximum Likelihood to do the estimation.

We make the distributional assumption that all errors are multivariate normal.

The test statistic we use to test whether m is MV efficient is a difference between the likelihood of two models, a constricted an unconstricted.

$$-2(\ell_T^c - \ell_T) = T(\ln|\widehat{V}_e^c| - ln|\widehat{V}_e|)$$

(here C signifies the constricted model)

Under the null this converges to a χ^2 distribution

Calculating *the* GRS statistic

The general expression in terms of likelihoods can be simplified substantially in the case of the CAPM with a risk free rate r_{ft}.

$$E[r_{it}] = r_{ft} - \beta_i(E[r_{mt} - r_{ft}])$$

The calculation can then be done in terms of *excess returns*, returns above the risk free rate.
This is what one usually calls *the* GRS statistic

Calculating *the* GRS statistic

Use the notation in chapter 5 of Campbell et al. [1997], and go through the construction of the GRS statistic.

Define Z_t as a $(N \times 1)$ vector of excess returns for N assets (or portfolios of assets). For these N assets, the excess returns can be described using the excess-return market model.

$$\mathbf{Z}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} Z_{mt} + \boldsymbol{\epsilon}_t$$

$$E[\boldsymbol{\epsilon}_t] = \mathbf{0}$$

$$E\left[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t{}'\right] = \boldsymbol{\Sigma}$$

$$E[Z_{mt} = \mu_m$$

$$E[(Z_{mt} - \mu_m)^2] = \sigma_m^2$$

$$\text{cov}(Z_{mt}, \boldsymbol{\epsilon}_t) = 0$$

$\boldsymbol{\beta}$ is the $(N \times 1)$ vector of betas, Z_{mt} is the time period t market portfolio excess return, and $\boldsymbol{\alpha}$ and $\boldsymbol{\epsilon}_t$ are $(N \times 1)$ vectors of asset return intercepts and disturbances, respectively.

The maximum likelihood estimates are

$$\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\mu}} - \hat{\beta}\hat{\mu}_m$$

$$\hat{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T}(\mathbf{Z}_t - \hat{\boldsymbol{\mu}})(Z_{mt} - \hat{\mu_m})}{\sum_{t=1}^{T}(Z_{mt} - \hat{\mu}_m)^2}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{Z} - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}}Z_{mt})(\mathbf{Z} - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}}Z_{mt})'$$

where

$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{Z}_t,$$

$$\hat{\mu}_m = \frac{1}{T}\sum_{t=1}^{T}Z_{mt}$$

and

$$\hat{\sigma}_m^2 = \frac{1}{T}\sum_{t=1}^{T}(Z_{mt} - \hat{\mu}_m)^2$$

These are the same as the OLS estimators.

We want to test the null hypothesis

$$\mathbf{H}_0 : \quad \boldsymbol{\alpha} = \mathbf{0}$$

agains the alternative

$$\mathbf{H}_A : \quad \boldsymbol{\alpha} \neq \mathbf{0}$$

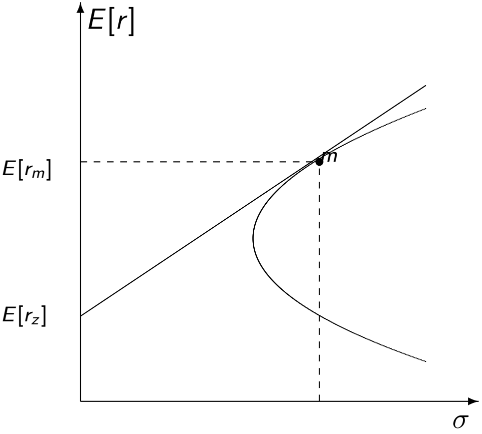
The GRS statistic J_i

$$J_1 = \frac{(T - N - 1)}{N} \left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2} \right]^{-1} \hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}$$

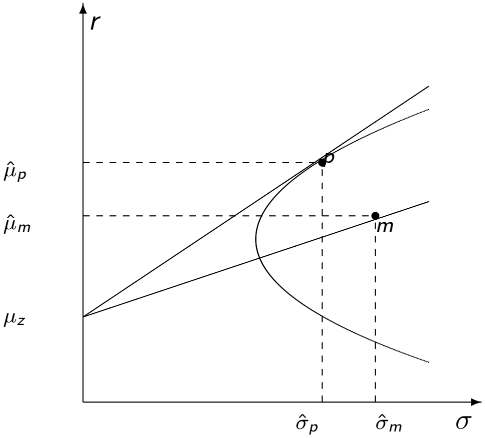
is under the null unconditionally distributed central F with N degrees of freedom in the numerator and $T - N - 1$ degrees of freedom in the denominator.

Geometric intuition

We are interested in a portfolio m. What we would like to know is whether m was on the MV frontier in the ex ante case:



In *ex post* MV space, we can always form the *ex post* efficient frontier using the actual portfolio m.



The test statistic measures the difference in the slope of the two lines.

If this difference is large, we think that the market portfolio is not ex ante efficient.

This is shown algebraically by Gibbons et al. [1989], who show that the GRS statistic J_1 can alternatively be calculated as

$$J_1 = \frac{(T - N - 1)}{N} \left(\frac{\frac{\hat{\mu}_q^2}{\hat{\sigma}_q^2} - \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}}{1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}} \right)$$

where the portfolio denoted by q denotes the *ex post* tangency portfolio constructed from the N included assets *plus* the market portfolio.

Example GRS calculation

Use "usual suspects" – the US portfolios provided by Ken French.

Exercise

One way to test the CAPM is to test whether the market portfolio is efficient. Let m denote a candidate for the market portfolio. Suppose that beside m there are N risky assets available. Suppose also that m is not a portfolio (linear combination) of these N assets.
Let $\tilde{r}_{it}$ denote the return of asset i in excess of the risk free rate. Consider the following regressions

$$\tilde{r}_{it} = \alpha_i + \beta_i r_{mt} + \epsilon_{it}$$

for all $1 \leq i \leq N$. Suppose further that conditional on $\tilde{r}_m$ the disturbance terms $\tilde{\epsilon}_{it}$ are jointly normally distributed with mean zero and nonsingular covariance matrix $\mathbf{\Sigma}$.

Let $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_N)$ denote the vector of intercept estimates from the previous regressions, and let $\hat{\boldsymbol{\Sigma}}$ denote the estimate of the covariance matrix. Furthermore, let $\hat{\theta}_m = \frac{\bar{r}_m}{s_m}$ denote the Sharpe ratio of portfolio m, where $\bar{r}_m$ and s_m^2 are the sample mean and variance of excess returns of portfolio m.
Gibbons, Ross and Shanken (1989) suggests the following statistic for testing the efficiency of portfolio m.

$$J_0 = T \frac{\hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \hat{\theta}_m^2}$$

which has an χ^2 distribution with degrees of freedom N.
Alternatively, the statistic with finite-sample correction:

$$J_1 = \frac{(T - N - 1)}{N} \frac{\hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \hat{\theta}_m^2}$$

which is F distributed with parameters $T - N - 1$ and N.

From Ken French' data library download monthly series for 10 industry portfolios, and monthly time-series of the market and the risk free return.
You want to test whether the market is efficient using the GRS test above.

1. Compute the Sharpe ratio of the market
2. Run linear regressions of the excess returns of each portfolio on the excess return on the market. Estimate the intercepts $\hat{\alpha}_i$ and the variance-covariance matrix $\hat{\boldsymbol{\Sigma}}$.
3. Check whether the matrix $\hat{\boldsymbol{\Sigma}}$ is nonsingular.
4. If it is, calculate the GRS and evaluate it.

Solution

Show code for Matlab in lecture notes, here only go through the results using the R commands and output

In the following we use data 1926:7 to 2014:12

Read the data

```
> source ("~/data/2015/french_data/read_industries.R")
> source ("~/data/2015/french_data/read_pricing_factors.R"
>
> eR   <- (FF10IndusEW - RF)/100.0
> eRm  <- RMRF/100.0
>
> head(eR)
            NoDur   Durbl   Manuf   Enrgy   HiTec   Telcm
1926(7)    0.0095  0.0381  0.0208 -0.0241  0.0199  0.0109  (
1926(8)    0.0503  0.0003  0.0189  0.0411  0.0236  0.0083 -(
1926(9)   -0.0020 -0.0357  0.0030 -0.0374  0.0063 -0.0001 -(
1926(10)  -0.0291 -0.0978 -0.0543  0.0235 -0.0604 -0.0131 -(
1926(11)   0.0576 -0.0069  0.0138  0.0128  0.0101  0.0117  (
1926(12)   0.0013  0.0452  0.0311  0.0464  0.0118  0.0134  (
```

Solution ctd

Sharpe Ratio for the market

```
> SharpeMarket <- mean(eRm)/sd(eRm)
> print(SharpeMarket)
[1] 0.1209972
```

Solution ctd

Regress each portfolio return on the market

```
> regr <- lm(eR~eRm)
>
> alpha <- as.matrix(regr$coefficients[1,])
> print(alpha)
                [,1]
NoDur  0.0020251126
Durbl -0.0001208441
Manuf  0.0018728056
Enrgy  0.0032662292
HiTec  0.0023540516
Telcm  0.0029028003
Shops  0.0017803260
Hlth   0.0041723995
Utils  0.0025095998
Other  0.0020429201
```

Solution ctd

The covariance matrix

```
> Sigma <- cov(as.matrix(regr$residuals))
> print(Sigma)
             NoDur        Durbl        Manuf        Enrgy
NoDur 0.0010366995 0.0010968373 0.0008440374 6.090463e-04
Durbl 0.0010968373 0.0020304898 0.0012219901 7.441056e-04
Manuf 0.0008440374 0.0012219901 0.0010780858 8.711581e-04
Enrgy 0.0006090463 0.0007441056 0.0008711581 3.146153e-03
HiTec 0.0007188427 0.0011128651 0.0008497387 4.452530e-04
...
```

Solution ctd

The inverse of the covariance matrix

```
> SigmaInv <- solve(Sigma)
> print(SigmaInv)
            NoDur       Durbl       Manuf       Enrgy
NoDur  4356.02257  -257.73017 -1392.22052   39.324788   383.
Durbl  -257.73017  1924.83122 -1426.77836  133.174808 -197.
Manuf -1392.22052 -1426.77836  5073.26833 -526.429312 -721.
Enrgy    39.32479   133.17481  -526.42931  450.537842    68.
HiTec   383.47433  -197.80170  -721.96819   68.336477 1360.
....
```

Solution ctd

Calculating the GRS statistics. Note the commands for matrix multiplication

```
> J0 <- T * ( t(alpha) %*% SigmaInv %*% alpha )
          / (1 + SharpeMarket^2)
> print(J0)
         [,1]
[1,] 28.94252
> J1 <- (T-N-1)/N * ( t(alpha) %*% SigmaInv %*% alpha )
          / (1 + SharpeMarket^2)
> print(J1)
         [,1]
[1,] 2.864274
```

Solution ctd

Testing for significance

```
> pchisq(J0,N,lower.tail=FALSE)
             [,1]
[1,] 0.001273022
> pf(J1,N,(T-N-1),lower.tail=FALSE)
             [,1]
[1,] 0.001576599
```

Both statistics reject the null.

Fisher Black, Michael Jensen, and Myron Scholes. The capital asset pricing model, some empirical tests. In Michael C Jensen, editor, *Studies in the theory of capital markets*. Preager, 1972.

John Y Campbell, Andrew W Lo, and A Craig MacKinlay. *The econometrics of financial markets*. Princeton University Press, 1997.

Michael R Gibbons. Multivariate tests of financial models, a new approach. *Journal of Financial Economics*, 10:3–27, March 1982.

Michael R Gibbons, Stephen A Ross, and Jay Shanken. A test of the efficiency of a given portfolio. *Econometrica*, 57:1121–1152, 1989.

A Craig MacKinlay. On multivariate tests of the CAPM. *Journal of Financial Economics*, 18:341–71, 1987.