Multivariate Tests of the CAPM under Normality

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Contents

1 Multivariate Tests of the CAPM

When we for example use the Black Jensen Approach, testing for $\alpha_i = 0$ in

$$r_{it} - r_{ft} = \alpha_i + (r_{mt} - r_{ft}) + \varepsilon$$

on an equation by equation basis, this is inefficient.

Want to aggregate the tests used in e.g. Black, Jensen, and Scholes (1972) into a single test statistic. If we are willing to make distributional assumptions, in this case multivariate normality, can use Maximum Likelihood methods to construct an aggregate test.

This was developed in a sequence of papers: Gibbons (1982), MacKinlay (1987) and Gibbons, Ross, and Shanken (1989).

We give an overview of the first and last of these papers.

2 The Gibbons (1982) paper, how to formulate the multivariate model

We use the Gibbons paper to show how to write the problem in matrix form, and construct a single system for testing.

Define

$$\tilde{R}_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ \vdots \\ R_{iT} \end{bmatrix}, \quad i_T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{R}_m = \begin{bmatrix} R_{m1} \\ R_{m2} \\ \vdots \\ R_{mT} \end{bmatrix}, \quad \text{and } \tilde{\eta}_i = \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{iT} \end{bmatrix}$$

η_i is the error term, and in the paper this is assumed independently, normally distributed (iidn).

$$\tilde{\eta}_i \sim \mathcal{N}(0, \sigma_{ii} I_T)$$

and we are looking at

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

This is a similar setup as in Black et al. (1972).

Note that this in the zero-beta varsion of the CAPM, which imposes

$$\tilde{R}_i = \tilde{r}_{zc} + \beta_i(\tilde{R}_m - \tilde{r}_{zc}) = \tilde{r}_{zc}(1-\beta_i) + \beta_i \tilde{R}_m$$

If the CAPM is true

$$\tilde{R}_i = \tilde{r}_{zc}(1-\beta_i) + \beta_i \tilde{R}_m$$

holds for all securities.

If we estimate

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

to test the CAPM, we test whether

$$\mathcal{H}_0 : \alpha_i = r_{zc}(1-\beta_i) \;\forall\; i$$

against

$$\mathcal{H}_A : \alpha_i \neq r_{zc}(1-\beta_i) \;\forall\; i$$

The problem is that we do not know r_{zc}, it must be estimated from the data. But then the estimation should take acount of that, under the null, r_{zc} is the same across securities. It is therefore helpful to stack the whole estimation into *one* set of equations.

We can stack the matrices in the following manner

$$\begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix} = \begin{bmatrix} \tilde{(}i_T : R_m) & 0 & \cdots & 0 \\ \tilde{0} & (i_T : R_m) & & \\ \vdots\vdots & & \ddots & \\ \tilde{0} & & & (i_T : R_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}$$

and

$$(i_T : R_m) = \begin{bmatrix} 1 & R_{m1} \\ 1 & R_{m2} \\ \vdots & \\ 1 & R_{mT} \end{bmatrix}$$

The system can be even more compactly written using Kroenecker products[1] as

$$\tilde{R}^* = [(i_T : R_m) \otimes I_N] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \tilde{\eta}^*$$

where we have defined

$$\eta^* = \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}$$

and

$$\tilde{R}^* = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix}$$

$(1 \times T \cdot N)$ vectors

The null hypothesis involves N variables $\alpha_1, \cdots, \alpha_N$.

Using the classical test statistics:

Wald: Estimate all of the α_i, β_i's. Then test

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LM: Estimate a single α, say α^*. Then test relaxation of

$$\alpha^* = \alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LR: Use both restricted and urestricted estimates, compare fit.

3 Multivariate test of the CAPM - Gibbons - Ross and Shanken (1989)

Gibbons et al. (1989) uses the setup of Gibbons (1982) to construct a test statistic to answer only one question, whether the market portfolio m mean variance efficient.

[1] Kroenecker product $\otimes$:

$$A = \begin{pmatrix} a_{11} & a_{21} & & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ & & \ddots & \\ a_{1n} & & & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{21} & & b_{p1} \\ b_{12} & b_{22} & & b_{p2} \\ & & \ddots & \\ b_{1q} & & & b_{pq} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{21}B & & a_{m1}B \\ a_{12}B & a_{22}B & & a_{m2}B \\ & & \ddots & \\ a_{1n}B & & & a_{mn}B \end{pmatrix}$$

A is a $mp \times nq$ matrix

3.1 How to test for aggregate MV efficiency

Let us first show intuitively how such a construction of a test statistic is done.

Consider the estimation of the two following models:

Unconstrained model

$$r_{jt} = \alpha_j + \beta_j r_{mt} + e_{jt}$$

Constrained model

$$r_{jt} = r_{zt}(1 - \beta_j) + \beta_j r_{mt} + e_{jt}$$

The constrained model is a special case of the unconstrained model.

If the CAPM is true, and m *is* MV efficient, the constrained model is the true model. Hence, our estimate of α_j in the unconstrained model should be approximately equal to $r_{zt}(1-\beta_j)$ (the intercept in the constrained model)

All the multivariate tests of MV efficiency does is to compare the fit of these two models. If the difference is large (according to some statistical metric), reject MV efficiency. Otherwise accept it.

The difference between the methods lies in how to measure the (statistical) difference in fit of the two models. Such test statistics relies on using Maximum Likelihood to do the estimation, and having made the distributional assumption that all errors are multivariate normal. Define:

$$r_t = \begin{bmatrix} r_{1t} \\ \vdots \\ r_{nt} \end{bmatrix} \quad \alpha_t = \begin{bmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{nt} \end{bmatrix} \quad \beta_t = \begin{bmatrix} \beta_{1t} \\ \vdots \\ \beta_{nt} \end{bmatrix} \quad \text{and } e_t = \begin{bmatrix} e_{1t} \\ \vdots \\ e_{nt} \end{bmatrix}$$

The model is then written as

$$r_t = \alpha_t + \beta_t r_{mt} + e_t$$

with the distributional assumption

$$e_t \sim N(\mathbf{0}, V_t)$$

where V_t is the covariance matrix $E[e_t e_t'] = V_t$.

We find the estimates by maximising the log-likelihood ℓ_T with respect to the parameters of interest.

$$\ell_T = -\left(\frac{NT}{2}\right)\ln(2\pi) - \frac{T}{2}\ln\left|\widehat{V}_e\right| - \frac{1}{2}\sum_{t=1}^{T} \hat{e}_t' \widehat{V}_e^{-1} \hat{e}_t$$

We calculate the same function, but now using the estimates $\widehat{V}_e^c$ from the restricted model

$$\ell_T^c = -\left(\frac{NT}{2}\right)\ln(2\pi) - \frac{T}{2}\ln\left|\widehat{V}_e^c\right| - \frac{1}{2}\sum_{t=1}^{T} e_t^{c\prime} (\widehat{V}_e^c)^{-1} e_t^c$$

The test statistic we use to test whether m is MV efficient is then

$$-2(\ell_T^c - \ell_T) = T(\ln|\widehat{V}_e^c| - ln|\widehat{V}_e|)$$

It can be shown that this converges to a χ^2 distribution, and we use this to make probability statements about the outcome.

3.2 The GRS statistic

The general expression in terms of likelihoods above can be simplified substantially in the case of the CAPM with a risk free rate r_{ft}.

$$E[r_{it}] = r_{ft} - \beta_i(E[r_{mt} - r_{ft}])$$

The calculation can then be done in terms of *excess returns*, returns above the risk free rate.

Let us use the notation in chapter 5 of Campbell, Lo, and MacKinlay (1997), and go through the construction of the GRS statistic.

Define Z_t as a $(N \times 1)$ vector of excess returns for N assets (or portfolios of assets). For these N assets, the excess returns can be described using the excess-return market model.

$$\mathbf{Z}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} Z_{mt} + \boldsymbol{\epsilon}_t$$

$$E[\boldsymbol{\epsilon}_t] = \mathbf{0}$$

$$E\left[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t{}'\right] = \boldsymbol{\Sigma}$$

$$E[Z_{mt} = \mu_m$$

$$E[(Z_{mt} - \mu_m)^2] = \sigma_m^2$$

$$\operatorname{cov}(Z_{mt}, \boldsymbol{\epsilon}_t) = 0$$

$\boldsymbol{\beta}$ is the $(N \times 1)$ vector of betas, Z_{mt} is the time period t market portfolio excess return, and $\boldsymbol{\alpha}$ and $\boldsymbol{\epsilon}_t$ are $(N \times 1)$ vectors of asset return intercepts and disturbances, respectively.

The maximum likelihood estimates are

$$\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\mu}} - \hat{\beta}\hat{\mu}_m$$

$$\hat{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T} (\mathbf{Z}_t - \hat{\boldsymbol{\mu}})(Z_{mt} - \hat{\mu_m})}{\sum_{t=1}^{T} (Z_{mt} - \hat{\mu}_m)^2}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} (\mathbf{Z} - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} Z_{mt})(\mathbf{Z} - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} Z_{mt})'$$

where

$$\hat{\boldsymbol{\mu}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{Z}_t,$$

$$\hat{\mu}_m = \frac{1}{T} \sum_{t=1}^{T} Z_{mt}$$

and

$$\hat{\sigma}_m^2 = \frac{1}{T} \sum_{t=1}^{T} (Z_{mt} - \hat{\mu}_m)^2$$

These are the same as the OLS estimators.

We want to test the null hypothesis

$$\mathbf{H}_0: \quad \boldsymbol{\alpha} = \mathbf{0}$$

agains the alternative

$$\mathbf{H}_A: \quad \boldsymbol{\alpha} \neq \mathbf{0}$$

The GRS statistic J_i

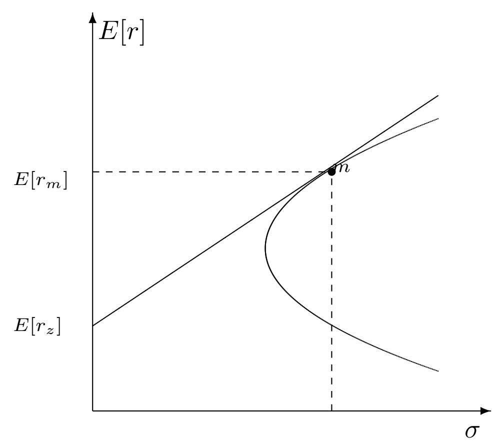
$$J_1 = \frac{(T - N - 1)}{N} \left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2} \right]^{-1} \hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}$$

is under the null unconditionally distributed central F with N degrees of freedom in the numerator and $T - N - 1$ degrees of freedom in the denominator.

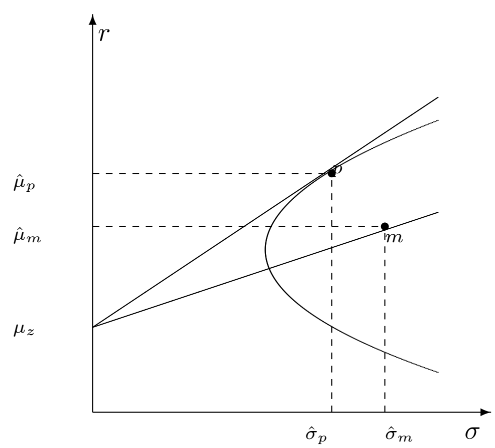
3.3 The Geometric Intuition of the GRS statistic

Let us look at some geometric intuition:

We are interested in a portfolio m. What we would like to know is whether m was on the MV frontier in the ex ante case:



In *ex post* MV space, we can always form the *ex post* efficient frontier:



Here m is the ex post outcome for the portfolio m and p is an ex post frontier portfolio. Intuitively, the test statistic measures the difference in the slope of the two lines in the picture. If this difference is large, we think that the market portfolio is not ex ante efficient.

This is shown algebraically by Gibbons et al. (1989), who show that the GRS statistic J_1 can alternatively be calculated as

$$J_1 = \frac{(T-N-1)}{N} \left(\frac{\frac{\hat{\mu}_q^2}{\hat{\sigma}_q^2} - \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}}{1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}} \right)$$

where the portfolio denoted by q denotes the *ex post* tangency portfolio constructed from the N included assets *plus* the market portfolio.

3.4 Example calculation of the GRS statistic

Exercise 1.

One way to test the CAPM is to test whether the market portfolio is efficient. Let m denote a candidate for the market portfolio. Suppose that beside m there are N risky assets available. Suppose also that m is not a portfolio (linear combination) of these N assets.

Let $\tilde{r}_{it}$ denote the return of asset i in excess of the risk free rate. Consider the following regressions

$$\tilde{r}_{it} = \alpha_i + \beta_i r_{mt} + \epsilon_{it}$$

for all $1 \leq i \leq N$. Suppose further that conditional on $\tilde{r}_m$ the disturbance terms $\tilde{\epsilon}_{it}$ are jointly normally distributed with mean zero and nonsingular covariance matrix $\mathbf{\Sigma}$.

Let $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_N)$ denote the vector of intercept estimates from the previous regressions, and let $\hat{\mathbf{\Sigma}}$ denote the estimate of the covariance matrix. Furthermore, let $\hat{\theta}_m = \frac{\overline{r}_m}{s_m}$ denote the Sharpe ratio of portfolio m, where $\overline{r}_m$ and s_m^2 are the sample mean and variance of excess returns of portfolio m.

Gibbons, Ross and Shanken (1989) suggests the following statistic for testing the efficiency of portfolio m.

$$J_0 = T \frac{\hat{\boldsymbol{\alpha}}' \hat{\mathbf{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \hat{\theta}_m^2}$$

which has an χ^2 distribution with degrees of freedom N.

Alternatively, the statistic with finite-sample correction:

$$J_1 = \frac{(T - N - 1)}{N} \frac{\hat{\boldsymbol{\alpha}}' \hat{\mathbf{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \hat{\theta}_m^2}$$

which is F distributed with parameters $T - N - 1$ and N.

From Ken French' data library download monthly series for 10 industry portfolios, and monthly time-series of the market and the risk free return.

You want to test whether the market is efficient using the GRS test above.

1. Compute the Sharpe ratio of the market
2. Run linear regressions of the excess returns of each portfolio on the excess return on the market. Estimate the intercepts $\hat{\alpha}_i$ and the variance-covariance matrix $\hat{\mathbf{\Sigma}}$.
3. Check whether the matrix $\hat{\mathbf{\Sigma}}$ is nonsingular.
4. If it is, calculate the GRS and evaluate it.

Solution to Exercise 1.

We show code for both octave and R

MATLAB:

```
Rets = dlmread("~/data/2015/french_data/10_Industry_Portfolios_monthly_ew.txt","",12,0);
Factors = dlmread ("~/data/2015/french_data/F-F_Research_Data_Factors_monthly.txt","",4,0);

RMRF = Factors(:,2);
Rf = Factors(:,5);
                                        # subtract Rf from each row
eR = Rets(:,2)-Rf;
eR =[eR Rets(:,3)-Rf];
eR =[eR Rets(:,4)-Rf];
eR =[eR Rets(:,5)-Rf];
eR =[eR Rets(:,6)-Rf];
eR =[eR Rets(:,7)-Rf];
eR =[eR Rets(:,8)-Rf];
```

```
eR =[eR Rets(:,9)-Rf];
eR =[eR Rets(:,10)-Rf];
eR =[eR Rets(:,11)-Rf];
                                  # challenge: find a more compact way
T = rows(eR)
N = columns(eR)

eR = eR/100.0;
eRm = RMRF/100.0;
[b1, sigma1, e1] = ols(eR(:,1),[ones(T,1) eRm] );
[b2, sigma2, e2] = ols(eR(:,2),[ones(T,1) eRm] );
[b3, sigma3, e3] = ols(eR(:,3),[ones(T,1) eRm] );
[b4, sigma4, e4] = ols(eR(:,4),[ones(T,1) eRm] );
[b5, sigma5, e5] = ols(eR(:,5),[ones(T,1) eRm] );
[b6, sigma6, e6] = ols(eR(:,6),[ones(T,1) eRm] );
[b7, sigma7, e7] = ols(eR(:,7),[ones(T,1) eRm] );
[b8, sigma8, e8] = ols(eR(:,8),[ones(T,1) eRm] );
[b9, sigma9, e9] = ols(eR(:,9),[ones(T,1) eRm] );
[b10, sigma10, e10] = ols(eR(:,10),[ones(T,1) eRm] );

alpha=[b1(1),b2(1),b3(1),b4(1),b5(1),b6(1),b7(1),b8(1),b9(1),b10(1)]';
e = [e1, e2, e3, e4, e5, e6, e7 ,e8, e9, e10];
Sigma = cov(e)
SigmaInv = inv(Sigma)
Sm = mean(eRm)/std(eRm)
J0 = T * inv(1+Sm^2)* (alpha')*SigmaInv*alpha
J1 = ((T-N-1)/N) * inv(1+Sm^2)*alpha'*SigmaInv*alpha
```

R:

```
source ("~/data/2015/french_data/read_industries.R")
source ("~/data/2015/french_data/read_pricing_factors.R")

eR  <- (FF10IndusEW - RF)/100.0
eRm <- RMRF/100.0
head(eR)
head(eRm)

T <- length(eRm)
N <- ncol(eR)
                                        # do all regressions at once
regr <- lm(eR~eRm)

alpha <- as.matrix(regr$coefficients[1,])
print(alpha)

Sigma <- cov(as.matrix(regr$residuals))
SigmaInv <- solve(Sigma)

SharpeMarket <- mean(eRm)/sd(eRm)
print(SharpeMarket)

J0 <- T * ( t(alpha) %*% SigmaInv %*% alpha ) / (1 + SharpeMarket^2)
print(J0)
J1 <- (T-N-1)/N * ( t(alpha) %*% SigmaInv %*% alpha ) / (1 + SharpeMarket^2)
print(J1)

pchisq(J0,N,lower.tail=FALSE)
pf(J1,N,(T-N-1),lower.tail=FALSE)
```

Let us go through the results using the R commands and output
In the following we use data 1926:7 to 2014:12
Read the data

```
> source ("~/data/2015/french_data/read_industries.R")
> source ("~/data/2015/french_data/read_pricing_factors.R")
>
> eR   <- (FF10IndusEW - RF)/100.0
> eRm  <- RMRF/100.0
>
> head(eR)
           NoDur   Durbl   Manuf   Enrgy   HiTec   Telcm   Shops    Hlth
1926(7)   0.0095  0.0381  0.0208 -0.0241  0.0199  0.0109  0.0047  0.0223
1926(8)   0.0503  0.0003  0.0189  0.0411  0.0236  0.0083 -0.0028  0.0586
1926(9)  -0.0020 -0.0357  0.0030 -0.0374  0.0063 -0.0001 -0.0094  0.0057
1926(10) -0.0291 -0.0978 -0.0543  0.0235 -0.0604 -0.0131 -0.0360 -0.0019
1926(11)  0.0576 -0.0069  0.0138  0.0128  0.0101  0.0117  0.0104  0.0689
1926(12)  0.0013  0.0452  0.0311  0.0464  0.0118  0.0134  0.0253  0.0014
           Utils   Other
1926(7)   0.0463 -0.0001
1926(8)  -0.0225  0.0428
1926(9)   0.0183 -0.0076
1926(10) -0.0330 -0.0283
1926(11)  0.0540  0.0054
1926(12)  0.0144  0.0095
> head(eRm)
 1926(7)  1926(8)  1926(9) 1926(10) 1926(11) 1926(12)
  0.0296   0.0264   0.0036  -0.0324   0.0253   0.0262
> T <- length(eRm)
> N <- ncol(eR)
```

Sharpe Ratio for the market

```
> SharpeMarket <- mean(eRm)/sd(eRm)
> print(SharpeMarket)
[1] 0.1209972
```

Regress each portfolio return on the market

```
> regr <- lm(eR~eRm)
>
> alpha <- as.matrix(regr$coefficients[1,])
> print(alpha)
               [,1]
NoDur  0.0020251126
Durbl -0.0001208441
Manuf  0.0018728056
Enrgy  0.0032662292
HiTec  0.0023540516
Telcm  0.0029028003
Shops  0.0017803260
Hlth   0.0041723995
Utils  0.0025095998
Other  0.0020429201
```

The covariance matrix

```
> Sigma <- cov(as.matrix(regr$residuals))
> print(Sigma)
             NoDur        Durbl        Manuf        Enrgy         HiTec
NoDur 0.0010366995 0.0010968373 0.0008440374 6.090463e-04  0.0007188427
Durbl 0.0010968373 0.0020304898 0.0012219901 7.441056e-04  0.0011128651
```

```
Manuf 0.0008440374 0.0012219901 0.0010780858 8.711581e-04  0.0008497387
Enrgy 0.0006090463 0.0007441056 0.0008711581 3.146153e-03  0.0004452530
HiTec 0.0007188427 0.0011128651 0.0008497387 4.452530e-04  0.0020306733
...
```

The inverse of the covariance matrix

```
> SigmaInv <- solve(Sigma)
> print(SigmaInv)
            NoDur       Durbl       Manuf       Enrgy      HiTec       Telcm
NoDur  4356.02257  -257.73017 -1392.22052   39.324788  383.47433 -168.123983
Durbl  -257.73017  1924.83122 -1426.77836  133.174808 -197.80170   21.631908
Manuf -1392.22052 -1426.77836  5073.26833 -526.429312 -721.96819  293.683299
Enrgy    39.32479   133.17481  -526.42931  450.537842   68.33648   16.827799
HiTec   383.47433  -197.80170  -721.96819   68.336477 1360.73352 -414.242475
....
```

Calculating the GRS statistics. Note the commands for matrix multiplication

```
> J0 <- T * ( t(alpha) %*% SigmaInv %*% alpha ) / (1 + SharpeMarket^2)
> print(J0)
         [,1]
[1,] 28.94252
> J1 <- (T-N-1)/N * ( t(alpha) %*% SigmaInv %*% alpha ) / (1 + SharpeMarket^2)
> print(J1)
         [,1]
[1,] 2.864274
```

Testing for significance

```
> pchisq(J0,N,lower.tail=FALSE)
            [,1]
[1,] 0.001273022
> pf(J1,N,(T-N-1),lower.tail=FALSE)
            [,1]
[1,] 0.001576599
```

Both statistics reject the null.

Readings The original papers
Gibbons (1982), MacKinlay (1987) and Gibbons et al. (1989)
The Article MacKinlay and Richardson (1991) also covers this material.
Textbook discussions
Huang and Litzenberger (1988) 10.17, 10.34-10.40
Campbell et al. (1997) chapter 5
Cochrane (2005) ch 12.1
Campbell (2018) Ch 3

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