Estimating Factor Risk Premia using GMM

Assume that expected return for a stock in excess of the risk free return in equilibrium can be expressed as

$$E[er^i] = \sum_j \lambda_j \beta_j^i \tag{1}$$

$E[er^i]$ is expected excess return for stock i
$j \in \{1, .., J\}$ the number of factors affecting returns,
β_j^i is the exposure to risk factor j for stock i
λ_j is the risk premium for risk factor j common to the whole market.

Want to estimate risk premia for one or more factors, and test whether a model can price a collection of assets.
Two steps.
The first step: Black et al. [1972] time series regressions

$$er_t^i = a^i + \sum_{j=1}^{J} \beta_j^i f_{jt} + \varepsilon_t^i \tag{2}$$

er_t^i is the excess return for stock i,
a^i a constant term,
β_j^i the estimated exposure to factor f_j of stock i.

In this estimation we do not use the restriction of constant risk premia across assets.
Next step: Estimate factor risk premia, and test whether the model is able to price stocks/portfolios correctly.
Given the estimates from (2) the risk premium linked to factor j can be estimated by a cross-sectional regression

$$er^i = \sum_{j=1}^{J} \lambda_j \beta_j^i + \varepsilon^i \tag{3}$$

λ_j is the risk premium of factor j.
Finally: perform statistical tests on λ_j to investigate whether the risk premia of the various factors are significantly different from zero.

In this section we show results of such estimations of the system

$$E[\mathbf{er}] = \alpha + \beta \mathbf{f}$$

and

$$E[\mathbf{er}] = \beta\lambda$$

The estimate of λ in these regression has an interpretation as the *risk premium* associated with that particular factor.

Example: The US cross section

We will use the "usual suspects," the Fama French portfolios, to illustrate.
We first look at the case of a one-factor model,

$$E[\mathbf{er}] = \alpha + \beta e R_m$$

$$E[\mathbf{er}] = \beta \lambda$$

First, estimate an OLS.

```
> library(gmm)
> source("~/data/2015/french_data/read_industries.R")
> source("~/data/2015/french_data/read_pricing_factors.R")

> eR <- (FF5IndusEW-RF)/100.0
> eRm <- RMRF/100.0
>                       # take intersection to align data
> data  <- merge(eR,eRm,all=FALSE)
> eR    <- as.matrix(data[,1:5])
> eRm   <- as.vector(data[,6])
>                       # start with ols regression
> reg  <- lm(eR~eRm)
```

	Model 1	Model 2	Model 3	Model 4	Model 5
(Intercept)	0.00	0.00*	0.00	0.00**	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
eRm	1.17***	1.23***	1.37***	1.05***	1.23***
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Adj. R^2	0.77	0.82	0.75	0.65	0.72
Num. obs.	1062	1062	1062	1062	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Do a GMM on corresponding problem

```
> X <- cbind(eR,eRm)
>                            # first estimate with GMM same
> g1 <- function (parms,X) {
+   a <- parms[1:5]
+   b <- parms[6:10]
+   mcond<-c()
+   for (i in 1:5){
+     e <- X[,i]- a[i]- b[i]*X[,6]
+     mcond <- cbind(mcond,e,e*X[,6])
+   }
+   return (mcond);
+ }
>                  # use ols coefficients as starting value
> t1   <- as.matrix(c(reg$coefficients[1,],
                      reg$coefficients[2,]))
```

Results of GMM estimation

```
> res <- gmm(g1,X,t1)
> summary(res)
Coefficients:
            Estimate     Std. Error   t value      Pr(>|t|)
Theta[1]     1.5276e-03   1.1536e-03   1.3242e+00   1.8544e-01
Theta[2]     1.9256e-03   9.8452e-04   1.9559e+00   5.0477e-02
Theta[3]     2.3665e-03   1.4208e-03   1.6656e+00   9.5800e-02
Theta[4]     4.1724e-03   1.4701e-03   2.8382e+00   4.5376e-03
Theta[5]     2.0429e-03   1.1971e-03   1.7065e+00   8.7911e-02
Theta[6]     1.1702e+00   4.6897e-02   2.4952e+01  2.0329e-137
Theta[7]     1.2295e+00   4.8822e-02   2.5182e+01  6.2577e-140
Theta[8]     1.3669e+00   2.9319e-02   4.6622e+01   0.0000e+00
Theta[9]     1.0514e+00   3.2394e-02   3.2455e+01  4.6229e-231
Theta[10]    1.2267e+00   6.5302e-02   1.8784e+01   1.0132e-78
```

Coefficients

```
> print(res$coefficients)
   Theta[1]    Theta[2]    Theta[3]    Theta[4]    Theta[5]    T
0.001527604 0.001925630 0.002366473 0.004172399 0.002042920 1.17
   Theta[7]    Theta[8]    Theta[9]   Theta[10]
1.229452865 1.366913679 1.051350202 1.226650377
```

Then do the estimation of the factor risk premia

```
> a <- res$coefficients[1:5]
> b <- res$coefficients[6:10]
>              # then estimate crossectional restriction
> g2 <- function (parms,X) {
+   lambda <-  parms[1];
+   mcond<-c()
+   for (i in 1:5){
+       e <- X[,i] - lambda * b[i]
+       mcond <- cbind(mcond,e)
+   }
+   return (mcond);
+ }
> lbound <- c(-0.5)
> ubound <- c(0.5)
> t2 <- c(0.01)
> res2 <- gmm(g2,X,t2,method="Brent",
         upper=ubound,lower=lbound)
```

	Factor Premia
λ_1	0.0088^{***}
	(0.0020)
Criterion function	427.4046
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

We thus have an estimate of the risk premium for the market.

Doing the same thing with 3 factors.
5 industry portfolios, 3 factors

	Factor Premia
λ_1	0.0077^{**}
	(0.0029)
λ_2	0.0031
	(0.0025)
λ_3	-0.0002
	(0.0013)
Criterion function	38.0204
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

10 industry portfolios, 3 factors

	Factor Premia
λ_1	0.0105^{***}
	(0.0021)
λ_2	-0.0014
	(0.0021)
λ_3	-0.0005
	(0.0019)
Criterion function	878.5109
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

49 industry portfolios, 3 factors.

Note that the number of observations is less, the 49 industries are only complete after 1969.

	Factor Premia
λ_1	0.0115^{***}
	(0.0020)
λ_2	-0.0037^{*}
	(0.0018)
λ_3	-0.0056^{**}
	(0.0018)
Criterion function	22397.8551
Num. obs.	546

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Fisher Black, Michael Jensen, and Myron Scholes. The capital asset pricing model, some empirical tests. In Michael C Jensen, editor, *Studies in the theory of capital markets*. Preager, 1972.