Estimation of factor premia using GMM

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1 Estimating risk premia in a factor setting

In a theoretical factor model one will assume that expected return for a stock in excess of the risk free return in equilibrium can be expressed as

$$E[er^i] = \sum_j \lambda_j \beta_j^i \tag{1}$$

where $E[er^i]$ is expected excess return for stock i, $j \in \{1, .., J\}$ the number of factors affecting returns, β_j^i is the exposure to risk factor j for stock i and λ_j is the risk premium for risk factor j common to the whole market.

There are various methods to estimate risk premia for one or more factors, and testing whether a model can price a collection of assets. The traditional method uses two steps. The first step is the method developed by Black, Jensen, and Scholes (1972), time series regressions of the type

$$er_t^i = a^i + \sum_{j=1}^{J} \beta_j^i f_{jt} + \varepsilon_t^i \tag{2}$$

where er_t^i is the excess return for stock i, a^i a constant term, and β_j^i the estimated exposure to factor f_j of stock i. The estimated factor exposures measures the sensitivity of the return of an asset to movements in the factors. When a factor is expressed as a return series, for example as the return of a portfolio of large companies less the return of a portfolio of small companies, the factor model can be tested by testing the restriction that all the constant terms, a^i, equals zero. If this is rejected the model is rejected.

In this estimation we do not use the restriction of constant risk premia across assets. The next step in the the two step procedure is therefore to estimate factor risk premia, and test whether the model is able to price stocks/portfolios correctly. Given the estimates from (2) the risk premium linked to factor j can be estimated by a cross-sectional regression

$$er^i = \sum_{j=1}^{J} \lambda_j \beta_j^i + \varepsilon^i \tag{3}$$

λ_j is the risk premium of factor j. Finally one will perform statistical tests on λ_j to investigate whether the risk premia of the various factors are significantly different from zero.

The estimate of λ in these regression has an interpretation as the *risk premium* associated with that particular factor.

1.1 Example: The US cross section

We will use the "usual suspects," the Fama French portfolios, to illustrate.

We first look at the case of a one-factor model,

$$E[\mathbf{er}] = \alpha + \beta e R_m$$

$$E[\mathbf{er}] = \beta \lambda$$

We first show the R code that we use to do the estimation:

```
library(gmm)
library(stargazer)
source("~/data/2015/french_data/read_industries.R")
source("~/data/2015/french_data/read_pricing_factors.R")
eR <- (FF5IndusEW-RF)/100.0
eRm <- RMRF/100.0
                              # take intersection to align data
data <- merge(eR,eRm,all=FALSE)
eR   <- as.matrix(data[,1:5])
eRm <- as.vector(data[,6])
                              # start with ols regression
reg  <- lm(eR~eRm)

X <- cbind(eR,eRm)
                              # first estimate with GMM same way as OLS,
g1 <- function (parms,X) {
  a <- parms[1:5]
  b <- parms[6:10]
  mcond<-c()
  for (i in 1:5){
    e <- X[,i]- a[i]- b[i]*X[,6]
    mcond <- cbind(mcond,e,e*X[,6])
  }
  return (mcond);
}
                              # use ols coefficients as starting values
t1   <- as.matrix(c(reg$coefficients[1,],reg$coefficients[2,]))
res <- gmm(g1,X,t1)
summary(res)
print(res$coefficients)
a <- res$coefficients[1:5]
b <- res$coefficients[6:10]
                              # then estimate the crossectional restriction
g2 <- function (parms,X) {
  lambda <- parms[1];
  mcond<-c()
  for (i in 1:5){
      e <- X[,i] - lambda * b[i]
      mcond <- cbind(mcond,e)
  }
  return (mcond);
}
lbound <- c(-0.5)
ubound <- c(0.5)
t2 <- c(0.01)
res2 <- gmm(g2,X,t2,method="Brent",upper=ubound,lower=lbound)
summary(res2)
```

We then go over the estimation in some detail, showing the interaction with R

First, estimate an OLS.

We use 5 industry portfolios from 1926–2014.

```
> library(gmm)
> source("~/data/2015/french_data/read_industries.R")
> source("~/data/2015/french_data/read_pricing_factors.R")

> eR <- (FF5IndusEW-RF)/100.0
> eRm <- RMRF/100.0
>                                          # take intersection to align data
> data  <- merge(eR,eRm,all=FALSE)
> eR    <- as.matrix(data[,1:5])
> eRm   <- as.vector(data[,6])
>                                          # start with ols regression
> reg  <- lm(eR~eRm)
```

	Model 1	Model 2	Model 3	Model 4	Model 5
(Intercept)	0.00	0.00^{*}	0.00	0.00^{**}	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
eRm	1.17^{***}	1.23^{***}	1.37^{***}	1.05^{***}	1.23^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Adj. R^2	0.77	0.82	0.75	0.65	0.72
Num. obs.	1062	1062	1062	1062	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Do a GMM on corresponding problem

```
> X <- cbind(eR,eRm)
>                                          # first estimate with GMM same way as OLS,
> g1 <- function (parms,X) {
+   a <- parms[1:5]
+   b <- parms[6:10]
+   mcond<-c()
+   for (i in 1:5){
+     e <- X[,i]- a[i]- b[i]*X[,6]
+     mcond <- cbind(mcond,e,e*X[,6])
+   }
+   return (mcond);
+ }
>                                          # use ols coefficients as starting values
> t1   <- as.matrix(c(reg$coefficients[1,],reg$coefficients[2,]))
```

Results of GMM estimation

```
> res <- gmm(g1,X,t1)
> summary(res)

Call:
gmm(g = g1, x = X, t0 = t1)


Method:  twoStep

Kernel:  Quadratic Spectral

Coefficients:
```

```
           Estimate     Std. Error   t value      Pr(>|t|)
Theta[1]    1.5276e-03   1.1536e-03   1.3242e+00   1.8544e-01
Theta[2]    1.9256e-03   9.8452e-04   1.9559e+00   5.0477e-02
Theta[3]    2.3665e-03   1.4208e-03   1.6656e+00   9.5800e-02
Theta[4]    4.1724e-03   1.4701e-03   2.8382e+00   4.5376e-03
Theta[5]    2.0429e-03   1.1971e-03   1.7065e+00   8.7911e-02
Theta[6]    1.1702e+00   4.6897e-02   2.4952e+01  2.0329e-137
Theta[7]    1.2295e+00   4.8822e-02   2.5182e+01  6.2577e-140
Theta[8]    1.3669e+00   2.9319e-02   4.6622e+01   0.0000e+00
Theta[9]    1.0514e+00   3.2394e-02   3.2455e+01  4.6229e-231
Theta[10]   1.2267e+00   6.5302e-02   1.8784e+01   1.0132e-78

J-Test: degrees of freedom is 0
                J-test                P-value
Test E(g)=0:    1.06133292185383e-30  *******

#############
Information related to the numerical optimization
Convergence code =  0
Function eval. =  413
Gradian eval. =  NA
```

Coefficients

```
> print(res$coefficients)
   Theta[1]    Theta[2]    Theta[3]    Theta[4]    Theta[5]    Theta[6]
0.001527604 0.001925630 0.002366473 0.004172399 0.002042920 1.170170748
   Theta[7]    Theta[8]    Theta[9]   Theta[10]
1.229452865 1.366913679 1.051350202 1.226650377
```

Then do the estimation of the factor risk premia

```
> a <- res$coefficients[1:5]
> b <- res$coefficients[6:10]
>                                         # then estimate the extra crossectional restriction
> g2 <- function (parms,X) {
+   lambda <-  parms[1];
+   mcond<-c()
+   for (i in 1:5){
+       e <- X[,i] - lambda * b[i]
+       mcond <- cbind(mcond,e)
+   }
+   return (mcond);
+ }
> lbound <- c(-0.5)
> ubound <- c(0.5)
> t2 <- c(0.01)
> res2 <- gmm(g2,X,t2,method="Brent",upper=ubound,lower=lbound)
```

	Factor Premia
λ_1	0.0088^{***}
	(0.0020)
Criterion function	427.4046
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

We thus have an estimate of the risk premium for the market.
Doing the same thing with 3 factors.
5 industry portfolios, 3 factors

	Factor Premia
λ_1	0.0077^{**}
	(0.0029)
λ_2	0.0031
	(0.0025)
λ_3	−0.0002
	(0.0013)
Criterion function	38.0204
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Note that only the market is significant here, the other two factors are not. This may be due both to the factors *not* being relevant, or that with five portfolios we do not have enough power. To investigate the latter alternative, redo the analysis with larger crossections. First 10 industry portfolios:
10 industry portfolios, 3 factors

	Factor Premia
λ_1	0.0105^{***}
	(0.0021)
λ_2	−0.0014
	(0.0021)
λ_3	−0.0005
	(0.0019)
Criterion function	878.5109
Num. obs.	1062

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

and then 49 industry portfolios, 3 factors.
Note that the number of observations is less, the 49 industries are only complete after 1969.

	Factor Premia
λ_1	0.0115^{***}
	(0.0020)
λ_2	-0.0037^{*}
	(0.0018)
λ_3	-0.0056^{**}
	(0.0018)
Criterion function	22397.8551
Num. obs.	546

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

With this large a crossection we do find that all the three fama french factors have significant risk premia.

References

Fisher Black, Michael Jensen, and Myron Scholes. The capital asset pricing model, some empirical tests. In Michael C Jensen, editor, *Studies in the theory of capital markets*. Preager, 1972.