Estimation of CAPM using GMM

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1 Testing the CAPM in a GMM setting

Consider the CAPM relationship, specified by the "moment condition"

$$E[er_{it}] = \beta_i E[er_{mt}]$$

where er_{it} is the excess return on an asset or a portfolio, and er_{mt} the excess return on the market portfolio.

One can alternatively specify the CAPM more generally as

$$E[r_{it}] = E[r_{zt}] + \beta_i(E[r_{mt}] - E[r_{zt}])$$

where we let r_{zt} be the return on a "zero covariance" portfolio.

The typical way of testing this relationship, the Black, Jensen, and Scholes (1972) method, estimates the corresponding regression

$$er_{it} = \alpha_i + \beta_i er_{mt} + \varepsilon_{it}$$

and tests whether $\alpha_i = 0$ on an equation by equation basis.

This is econometically wasteful, since the restriction on the constant will hold for all assets, one want to test this jointly, in a multivariate setting. Doing this was proposed by Gibbons (1982), who showed how one could construct a multivariate statistic for testing this. His method was later expanded upon by Gibbons, Ross, and Shanken (1989). These statistic were developed under distributional assumptions that allowed us to use Maximum Likelihood, namely multivariate normality. What if these are not fulfilled, can we still construct a similar test statistic? This is done in the paper of MacKinlay and Richardson (1991) (MR). They construct a test statistic that essentially tests the same restriction, that $\alpha_I = 0$ or that $\alpha_i = E[r_{zt}](1 - \beta_i)$, but in a GMM framework, not a ML.

The setup is as follows.

Again, we have the usual regression

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}$$

We assume that

$$E[\varepsilon_{it}|r_{mt}] = 0$$

This implies two moment restrictions for each asset i:

$$E[\varepsilon_{it}] = E[(r_{it} - \alpha_i - \beta_i r_{mt})] = 0$$

$$E[\varepsilon_{it} r_{mt}] = E[(r_{it} - \alpha_i - \beta_i r_{mt}) r_{mt}] = 0$$

The model is exactly identified. We can "stack" these moment conditions and estimate the parameters $\{\alpha_i, \beta_i\}$ of the model, by the usual formulation using sample moments.

The tests discussed in the paper are different ways of testing the parametric restriction $\alpha_i = 0$.

We will show how the GMM framework can be used to test the CAPM.

The data we apply it to is 5 US industries provided by Ken French. We use data 1990-2019.

First, the CAPM estimated on an equation by equation basis, as illustrated below.

	Dependent variable:				
	eRi[, 1]	eRi[, 3]		eRi[, 4]	eRi[, 5]
	Cnsmr	Manuf	HiTec	Hlth	Other
	(1)	(2)	(3)	(4)	(5)
eRm	1.167***	1.372***	1.372***	1.061***	1.123***
	(0.023)	(0.031)	(0.031)	(0.033)	(0.024)
Constant	−0.002*	−0.0001	−0.0001	0.002	−0.001
	(0.001)	(0.002)	(0.002)	(0.002)	(0.001)
Observations	745	745	745	745	745
Adjusted R^2	0.773	0.721	0.721	0.579	0.744

Note: *p<0.1; **p<0.05; ***p<0.01

When doing this as a GMM estimation, the parameter estimates will be the same.

	Model 1
Cnsmr_(Intercept)	−0.002
	(0.001)
Manuf_(Intercept)	−0.001
	(0.001)
HiTec_(Intercept)	−0.000
	(0.002)
Hlth_(Intercept)	0.002
	(0.002)
Other_(Intercept)	−0.001
	(0.001)
Cnsmr_eRm	1.167***
	(0.056)
Manuf_eRm	1.202***
	(0.054)
HiTec_eRm	1.372***
	(0.035)
Hlth_eRm	1.061***
	(0.056)
Other_eRm	1.123***
	(0.075)
Criterion function	0.000
Num. obs.	745

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

But now we can do a joint test of whether all the intercepts are equal to zero

Table 1

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
Res.Df	2	745.500	3.536	743	744.2	746.8	748
Df	1	5.000		5.000	5.000	5.000	5.000
F	1	1.272		1.272	1.272	1.272	1.272
Pr(>F)	1	0.274		0.274	0.274	0.274	0.274

The below is the complete `R` code for doing this (except for the input routines for the French data).

```
library(gmm)
library(car)
library(texreg)
library(stargazer)
                                    # illustrate the use of two packages for writing to tables
outdir <- "../../results/2020_09_gmm_us/"
source("/home/bernt/data/2020/french_us_data/read_5_industries.R")
source("/home/bernt/data/2020/french_us_data/read_3_pricing_factors.R")
                                    # then take 5 portfolios, start in 1990
industries <- names(FF5IndusEW)
eRindus <-FF5IndusEW-RF
eRi <- window(eRindus,start=c(1990,1))
data <- merge(eRi,RMRF,all=FALSE)
                                    # estimate as separate linear regressions
eRi <- as.matrix(data[,1:5])
eRm <- as.matrix(data[,6])
summary(lm(eRi~eRm))
results <- list(lm(eRi[,1]~eRm),lm(eRi[,3]~eRm),lm(eRi[,3]~eRm),lm(eRi[,4]~eRm),lm(eRi[,5]~eRm))
ofilename <- paste0(outdir,"linear_regressions_5_industries.tex")
stargazer(results,
          out=ofilename,
          column.labels=industries,

          float=FALSE,
          omit.stat=c("rsq","f","ser"))

                                    # estimate as a joint gmm system
res   <- gmm(eRi~eRm,x=eRm)
summary(res)
ofilename <- paste0(outdir,"gmm_5_industries.tex")
texreg(res,
       table=FALSE,
       digits=3,
       file=ofilename)

                                    # test the joint hypothesis that all intercepts are zero
R <- cbind(diag(5),matrix(0,5,5))
c  <- rep(0,5)
hyptest <- linearHypothesis(res,R,c,test="F")
ofilename <- paste0(outdir,"gmm_5_industries_hypothesis_test.tex")
stargazer(hyptest,
          out=ofilename,
          header=FALSE,
          omit.stat=c("rsq","f","ser"))
```

References

Fisher Black, Michael Jensen, and Myron Scholes. The capital asset pricing model, some empirical tests. In Michael C Jensen, editor, *Studies in the theory of capital markets*. Preager, 1972.

Michael R Gibbons. Multivariate tests of financial models, a new approach. *Journal of Financial Economics*, 10:3–27, March 1982.

Michael R Gibbons, Stephen A Ross, and Jay Shanken. A test of the efficiency of a given portfolio. *Econometrica*, 57:1121–1152, 1989.

A Craig MacKinlay and Matthew P Richardson. Using generalized method of moments to test mean-variance efficiency. *Journal of Finance*, 46:511–27, 1991.