# Matrix handlers (matlab, scilab, octave, etc)

Purpose: Manipulating numerical relationships using notation similar to linear algebra.

Usually work on collections of numbers, represented by vectors and/or matrices.

*Name* the objects of interest (vectors/matrices) and specify manipulations of these.

Create a vector:

```
v=[1 ; 1]
v =
   1
   1
```

Create a matrix:

```
>> A = [1 1;0 1]
A =
   1   1
   0   1
```

Multiply the two

```
>> A*v
ans =
   2
   1
```

# Finance examples: Term Structure of Interest rates

The term structure of interest rates is a relationship between time
to maturity and prices of dated future payments.

*Example*

A two-year Treasury bond with a face value of 1000 and an annual
coupon payment of 8% sells for 982.50. A one-year T bill, with a
face value of 100, and no coupons, sells for 90. Compounding is
discrete, annual. Given these market prices,

1. Find the prices $d(0, 1)$ and $d(0, 2)$ of one dollar received
   respectively one and two years from now.
2. Find the corresponding interest rates.

1. Discount factors (prices):

$$\begin{bmatrix} 982.50 = d(0,1)80 + d(0,2)1080 \\ 90 = d(0,1)100 \end{bmatrix}$$

Solving these equations we find prices

$$\left[ \begin{array}{l} d(0,1) = 0.9 \\ d(0,2) = 0.84 \end{array} \right]$$

and interest rates

$$\left[ \begin{array}{l} r(0,1) = 11\% \\ r(0,2) = 9\% \end{array} \right]$$

How is this example calculated with a matrix tool?

Define the vector of discount factors

$$\mathbf{d} = \left[ \begin{array}{c} d(0,1) \\ d(0,2) \end{array} \right]$$

the set of cash flows

$$\mathbf{C} = \left[ \begin{array}{cc} 80 & 1080 \\ 100 & 0 \end{array} \right]$$

and the set of bond prices

$$\mathbf{B} = \left[ \begin{array}{c} 982.50 \\ 90 \end{array} \right]$$

The set of bond prices satisfies

$$\mathbf{B} = \mathbf{C}\mathbf{d}'$$

which means we can find $\mathbf{d}$ as

$$\mathbf{d}' = \mathbf{C}^{-1}\mathbf{B}$$

$$\mathbf{d}' = \mathbf{C}^{-1}\mathbf{B} = \begin{bmatrix} 80 & 1080 \\ 100 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 982.50 \\ 90 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.84 \end{bmatrix}$$

```
> C=[80 1080; 100 0 ]
C =
    80   1080
   100      0
2> B=[982.5;90]
B =
  982.500
   90.000
> d=inv(C)*B
d =
  0.90000
  0.84306
```

You have estimated spot interest rates as follows:

| Year | Spot Rate, Percent |
|------|--------------------|
| 1 | $r(0,1) = 5.00$ |
| 2 | $r(0,2) = 5.40$ |
| 3 | $r(0,3) = 5.70$ |
| 4 | $r(0,4) = 5.90$ |
| 5 | $r(0,5) = 6.00$ |

Compounding is discrete, annual.

1. What are the discount factors for each date (that is, the present value of $1 paid in year $t$)?

2. What are the forward rates for each period?

3. Calculate the prices of the following Treasury notes. Assume they pay interest once a year, and have face values of 100.

   3.1 5 percent, 2-year note
   3.2 5 percent, 5-year note
   3.3 10 percent, 5-year note

4. Compare the duration of the two five year bonds.

Discount factors and forward prices

| Year | $r(0, t)$ | $d(0, t)$ | $f(0, t-1, t)$ |
|------|-----------|-----------|----------------|
| 1 | 5.0 | $\frac{1}{(1.05)} = 0.952$ | 5.0% |
| 2 | 5.4 | $\frac{1}{(1.054)^2} = 0.900$ | $\frac{(1.054)^2}{(1.05)} - 1 = 5.8\%$ |
| 3 | 5.7 | $\frac{1}{(1.057)^3} = 0.846$ | $\frac{(1.057)^3}{(1.05)(1.058)} - 1 = 6.3\%$ |
| 4 | 5.9 | $\frac{1}{(1.059)^4} = 0.795$ | $\frac{(1.059)^4}{(1.05)(1.058)(1.063)} - 1 = 6.5\%$ |
| 5 | 6.0 | $\frac{1}{(1.06)^5} = 0.747$ | $\frac{(1.06)^5}{(1.05)(1.058)(1.063)(1.065)} - 1 = 6.4\%$ |

Prices of coupon bonds, assuming a face value of 100.

1. 5%, 2 years:

$$PV = 0.05 \cdot 0.952 + 1.05 \cdot 0.9 = 99.279$$

2. 5%, 5 years:

$$PV = 0.05 \cdot 0.952 + 0.05 \cdot 0.9 + 0.05 \cdot 0.846 + 0.05 \cdot 0.795 + 1.05 \cdot 0.74 = 95.$$

3. 10%, 5 years

$$PV = 0.10 \cdot 0.952 + 0.10 \cdot 0.9 + 0.10 \cdot 0.846 + 0.10 \cdot 0.795 + 1.10 \cdot 0.747 = 11$$

Now, here is how this is done in a matrix tool:
Discount factors:

```
> r=[5 5.4 5.7 5.9 6]/100
r =
  0.050000  0.054000  0.057000  0.059000  0.060000
> t=1:5
t =
  1  2  3  4  5
> d=(1+r).^(-t)
d =
  0.95238  0.90016  0.84679  0.79509  0.74726
> f(1)=r(1);
> for i=2:5
> f(i) = ((1+r(i))^i)/((1+r(i-1))^(i-1))-1;
> endfor
f =
  0.050000  0.058015  0.063026  0.065023  0.064009
```

Calculating bond prices:

```
> b=[5 105 0 0 0]*d'
b = 99.279
> b=[5 5 5 5 105]*d'
b = 95.934
>b=[10 10 10 10 110]*d'
b = 117.14
```

Calculating duration

```
> C1=[5 5 5 5 105]
C1 =
    5    5    5    5   105
> B1=C1*d'
B1 = 95.934
> Dur1 = 1/B1 * (t.*C1)*d'
D1 = 4.5310
> C2=[10 10 10 10 110]
C2 =
   10   10   10   10   110
> B2 = C2*d'
B2 = 117.14
> Dur2 =  1/B2 * (t.*C2)*d'
Dur2 = 4.2318
```

Note the use of the element by element multiplication .* in this calculation.

*Example*
A 10%, two year bond is traded at a price of 90. The current one year spot rate is $r(0,1) = 12\%$ (with discrete, annual compounding).

1. Determine the duration and convexity of the bond, using both the full term structure and the Macaulay style calculations.

First need to find the two year spot rate

$$90 = 10d_1 + 110d_2$$

$$d_1 = \frac{1}{1 + 0.12} = 0.89286$$

$$d_2 = \frac{90 - 10d_1}{110} = 0.73701$$

$$r_2 = 0.16483$$

Here are some of the calculation in a matrix tool

```
> C=[10 110]
C =
   10  110
> r(1)=0.12
r =   0.12000
> d(1)=1/(1+r(1))
d =   0.89286
> d(2)=(90-10*d(1))/110
d =
  0.89286  0.73701
> d=d'
d =
  0.89286
  0.73701
> BondPrice=C*d
BondPrice = 90
> r(2)=d(2)^(-1/2)-1
r =
  0.12000  0.16483
```

We calculate duration using the two definitions

```
> Duration=1/BondPrice * ( 1*d(1)*C(1) + 2*d(2)*C(2))
Duration = 1.9008
> Duration=1/BondPrice * ( 1*C(1)/(1+y) + 2*C(2)/((1+y)^2)
Duration = 1.9044
```

Using the term structure we find duration as

$$D = 1.9008$$

using the Macaulay definition we find

$$D = 1.9044$$

Thus, not a major difference.

We also calculate the convexity for the two definitions

```
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*d(1)*C(1) + (2+2^2)*
Cx = 4.1463
> Cx=1/BondPrice * 1/(1+y)^2 * ( (1+1)*C(1)/(1+y) + (2+2^2)
Cx = 4.1570
```

Again, not a major difference with the two methods of calculating

Consider a second version of the matrix calculations, with slighly more compact use of notation in the calculations.

```
> C=[10 110]
C =
    10   110
> r=[0.12]
r = 0.12000
> d=(1+r)^(-1)
d = 0.89286
> d(2)=(90-10*d(1))/110
d =
  0.89286   0.73701
> r=d.^(-1./t)-1
r =
 0.12000   0.16483
> B=C*d'
B = 90
```

```
> y=irr([-B C])
y = 0.16249
8> D=1/B*(t.*C)*d'
D = 1.9008
> DMc = 1/B* (t.*C)*((1+y).^(-t))'
DMc = 1.9044
> Cx = 1/B*(1/(1+y))*((t+t.^2).*C)*d'
Cx = 4.8200
> Cx = 1/B*(1/(1+y))*((t+t.^2).*C)*((1+y).^(-t))'
Cx = 4.8324
```