Testing the CAPM one factor model - overview

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1 Classical Empirical tests of the CAPM.

We cover the testing of the Capital Asset Pricing Model. The perspective is the historical evolution of these tests, starting with the classical papers and then looking at some of the important improvements in the testing methodology employed.

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  - (Huang and Litzenberger, 1988, 10.34–10.40).
  - Gibbons, Ross, and Shanken (1989).
  - MacKinlay and Richardson (1991). GMM estimation
- Questioning the testability of the CAPM. The Roll Critique: Unobservability of the market portfolio.

– See (Huang and Litzenberger, 1988, 10.11,10.12)
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2 Introductory.

CAPM:

$$\begin{aligned} E_{t-1}[r_{it} - r_{zt}] &= \frac{\text{cov}_{t-1}(r_{pt}, r_{it})}{\text{var}_{t-1}(r_{pt})} E_{t-1}[r_{pt} - r_{zt}] \\ &= \beta_{it} E_{t-1}[r_{pt} - r_{zt}] \end{aligned}$$

Let us start by listing some of the major *conceptual* problems in testing the CAPM. (See H&L 10.8.)

- The CAPM is an ex ante relation, not ex post. (It only holds in expectation).

  Assume rational expectations, then observations are drawn from the ex ante distribution.

- CAPM is a static model, but we have a time series of data.

  We solve this by assuming that the CAPM holds period-by-period. See the relation between conditional and unconditional restrictions below.

- The market portfolio is unobservable. See section on Roll Critique below.

3 The relation between the conditional and unconditional versions of the CAPM.

Recall the CAPM in conditional form

$$E_{t-1}[r_{it} - r_{zt}] = \beta_{it} E_{t-1}[r_{pt} - r_{zt}] \tag{1}$$

or, written explicitly

$$E[r_{it} - r_{zt}|\Omega_{t-1}] = \beta_{it} E[r_{pt} - r_{zt}|\Omega_{t-1}]$$

This is assumed to hold over all possible information sets Ω_{t-1}. By taking expectations over Ω_{t-1}, we find that

$$E[E[r_{it} - r_{zt}|\Omega_{t-1}]] = E[\beta_{it} E[r_{pt} - r_{zt}|\Omega_{t-1}]]$$

which implies that in unconditional expectations, the following is the case

$$E[r_{it} - r_{zt}] = \beta_{it} E[r_{pt} - r_{zt}] \tag{2}$$

This fact is used as a justification for looking at the unconditional model. In the material we discuss today, we will only be looking at estimation where we assume the unconditional version of the model holds.[1]

Note that the conditional condition (1) implies the unconditional condition (2), but the opposite is not true, the unconditional CAPM may be true, but not the conditional. To see this, consider a simple counterexample:

Suppose at time $t-1$ an event δ_{t-1} is revealed,

$$\delta_{t-1} = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

This is known at time $t-1$, and δ_{t-1} influences the return in period t in the following way:

$$r_{pt} = 0.1\delta_{t-1} + \varepsilon_t$$

[1] We will return to the conditional (time-varying) model.

ε_t is independent of δ_{t-1} and has mean zero. Then the conditional CAPM will be different depending on the realization of δ_{t-1}:

$$r_{pt} = \begin{cases} 0.1 + \varepsilon_t \\ -0.1 + \varepsilon_t \end{cases}$$

and

$$E[r_{pt}] = \begin{cases} 0.1 \\ -0.1 \end{cases}$$

with the conditional CAPM relations

$$E[r_{it} - r_{zt}|\delta_t] = \begin{cases} \beta_{ip} \cdot (0.1 - E[r_{zt}|\delta_{t-1} = 1]) \\ \beta_{ip} \cdot (-0.1 - E[r_{zt}|\delta_{t-1} = -1]) \end{cases}$$

So, even if the unconditional expectation holds

$$E[r_{it} - r_{zt}] = \beta_{ip}(E[r_{pt} - r_{zt}])$$

this will not guarantee that the each of the conditional expectations hold...

4 Testable implications of the CAPM.

Consider the CAPM relation in the usual form:

$$E[r_{it} - r_{zt}] = \beta_{it} E[r_{mt} - r_{zt}]$$

We will often write this in *excess return form* by assuming the zero-beta or risk-free return is subtracted. Which of these formulations we are considering will as a rule be obvious from the discussion.

$$E[r_{it}] = \beta_{it} E[r_{mt}]$$

What are the testable implications of this relation?

1. m is mean-variance efficient.

   To test this, consider

   $$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}$$

   Impose

   $$E[\varepsilon_{it}|r_{mt}] = 0$$

   We can test MV-efficiency by testing whether $\alpha = 0$ if the above is in excess return form, or testing $\alpha_i = (1 - \beta_i) r_{zt}$ if the model is in return form.

2. β_{it} is positively related to $E[r_{it}]$, which we can test by testing whether $E[r_m] > 0$ if the model is in excess return form, or testing whether $E[r_{mt} - r_{zt}] > 0$.

   In other words, we are testing if the risk premium is positive.

5 Classical tests.

One difference between the classical tests and the newer ones too keep in mind is that the classical ones tend to look at one return in isolation, whereas the newer ones try to aggregate all the returns into one test statistic.

5.1 The Fama and MacBeth (1973) paper

5.1.1 Short summary

This paper uses the Crossectional relation

$$(r_{jt} - r_{ft}) = a_t + b_t \beta_{j\hat{m}} + u_{jt} \; j = 1, 2, \ldots, N$$

Compare to the CAPM

$$r_{jt} - r_{ft} = (r_{mt} - r_{ft})\beta_{jm}$$

Prediction of the CAPM:

$$E[a_t] = 0$$

$$E[b_t] = (E[r_m] - r_f) > 0$$

To test this, average estimated a_t, b_t:

Test whether

$$E[a_t] = 0, \quad \frac{1}{T}\sum_{t=1}^{T} a_t \to 0$$

$$E[b_t] > 0, \quad \frac{1}{T}\sum_{t=1}^{T} b_t > 0$$

To do these tests we need an estimate of $\beta_{j\hat{m}}$. The "usual" approach is to use time series data to estimate $\beta_{j\hat{m}}$ from the "market model"

$$r_{jt} = \alpha_j + \beta_{jm} r_{mt} + \varepsilon_{jt}$$

on data *before* the crossection.

Results usually $\bar{a} > 0$ and $\bar{b} > 0$.

5.2 The Black et al. (1972) approach

Time series regressions.

$$r_{jt} = \alpha_j + \beta_{jm} r_{mt} + \varepsilon_{jt}$$

CAPM imposes

$$\alpha_j = r_{zc}(1 - \beta_{jm})$$

Why?

$$r_{jt} = r_{zc} + (r_m - r_{zc})\beta_{jm} = r_{zc} - \beta_{jm} r_{zc} + \beta_{jm} r_{mt} = r_{zc}(1 - \beta_{jm}) + \beta_{jm} r_{mt}$$

To test the CAPM, test whether

$$E\left[\frac{\alpha_j}{1 - \beta_{jm}}\right] = r_{zc}, \text{ or } \frac{1}{N}\sum_{i=1}^{N} \frac{\alpha_j}{1 - \beta_{jm}} = r_{zc}$$

This is inefficient because the estimation is by company first, before the test is performed.

5.3 Summarizing classical work

A number of early test used a similar setup, and got similar results, well known examples are Blume and Friend (1973), Fama (1976) and Miller and Scholes (1972).

Usually, the evidence of positivity of $E[R_{it}] - E[R_{zt}]$ was viewed as evidence in favour of a constrained borrowing version of the CAPM, not necessarily a rejection of the model.

What are the problems with the classical tests?

Let us first look at two econometric problems, discussed in comprehensive detail in H&L 10.15 to 10.35.

Consider the equation

$$r_{it} = a_{it} + b_{it}\beta_i + \varepsilon_{it}$$

The two econometric issues that are addressed in the context of the classical framework are

- We may have dependencies in ε_{it}. Under reasonable types of dependencies, OLS estimates are consistent estimates of the coefficient, but not of the variance. Replace OLS with other estimators to get efficient and consistent test statistics.
- The β coefficients above are not known, they must be replaced with estimates. This is termed an "Errors in Variables" situation. OLS is then not even consistent. To solve
  - Group data in a way that reduces the measurement error in the β estimates.
  - Use an instrumental variables (IV) approach. Try to find instruments unrelated to the measurement error in the estimation.
  - Adjusted Generalised Least Squares (GLS). Take into account the variance from the β estimation in estimating b.

6 Multivariate tests in a normal setting

Want to aggregate the tests used in e.g. Black et al. (1972) into a single test statistic. This was developed in a sequence of papers Gibbons (1982), MacKinlay (1987) and Gibbons et al. (1989).

We give an overview of the first and last of these papers.

6.1 The Gibbons (1982) paper, how to formulate the multivariate model.

The main problem with looking at the multivariate model is notational. How do we write the model in matrix form?

If we define

$$\tilde{R}_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ \vdots \\ R_{iT} \end{bmatrix}, \quad i_T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{R}_m = \begin{bmatrix} R_{m1} \\ R_{m2} \\ \vdots \\ R_{mT} \end{bmatrix}, \quad \text{and } \tilde{\eta}_i = \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{iT} \end{bmatrix}$$

η_i is the error term, and in the paper this is assumed independently, normally distributed (iidn).

$$\tilde{\eta}_i \sim \mathcal{N}(0, \sigma_{ii} I_T)$$

and we are looking at

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

This is the same setup as in Black et al. (1972), and we have seen that this imposes

$$\tilde{R}_i = \tilde{r}_{zc} + \beta_i(\tilde{R}_m - \tilde{r}_{zc}) = \tilde{r}_{zc}(1 - \beta_i) + \beta_i \tilde{R}_m$$

If the CAPM is true

$$\tilde{R}_i = \tilde{r}_{zc}(1 - \beta_i) + \beta_i \tilde{R}_m$$

holds for all securities.

If we estimate

$$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_m + \tilde{\eta}_i$$

to test the CAPM, we test whether

$$\mathcal{H}_0 : \alpha_i = r_{zc}(1 - \beta_i) \ \forall \ i$$

against

$$\mathcal{H}_A : \alpha_i \neq r_{zc}(1 - \beta_i) \ \forall \ i$$

The problem is that we do not know r_{zc}, it must be estimated from the data. But then the estimation should take acount of that, under the null, r_{zc} is the same across securities. It is therefore helpful to stack the whole estimation into *one* set of equations.

We can stack the matrices in the following manner

$$\begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix} = \begin{bmatrix} \tilde{(i_T : R_m)} & 0 & \cdots & 0 \\ \tilde{0} & (i_T : R_m) & & \\ \vdots\vdots & & \ddots & \\ \tilde{0} & & & (i_T : R_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}$$

and

$$(i_T : R_m) = \begin{bmatrix} 1 & R_{m1} \\ 1 & R_{m2} \\ \vdots & \\ 1 & R_{mT} \end{bmatrix}$$

The system can be even more compactly written using Kroenecker products[2] as

$$\tilde{R}^* = [(i_T : R_m) \otimes I_N] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_N \\ \beta_N \end{bmatrix} + \tilde{\eta}^*$$

where we have defined

$$\eta^* = \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}$$

and

$$\tilde{R}^* = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \\ \vdots \\ \tilde{R}_N \end{bmatrix}$$

$(1 \times T \cdot N)$ vectors

The null hypothesis involves N variables $\alpha_1, \cdots, \alpha_N$.

Using the classical test statistics:

[2]Kroenecker product $\otimes$:

$$A = \begin{pmatrix} a_{11} & a_{21} & & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ & & \ddots & \\ a_{1n} & & & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{21} & & b_{p1} \\ b_{12} & b_{22} & & b_{p2} \\ & & \ddots & \\ b_{1q} & & & b_{pq} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{21}B & & a_{m1}B \\ a_{12}B & a_{22}B & & a_{m2}B \\ & & \ddots & \\ a_{1n}B & & & a_{mn}B \end{pmatrix}$$

A is a $mp \times nq$ matrix

Wald: Estimate all of the α_i, β_i's. Then test

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LM: Estimate one α_i, say α^*. Then test relaxation of

$$\alpha^* = \alpha_1 = \alpha_2 = \cdots = \alpha_N$$

LR: Use both restricted and urestricted estimates, compare fit.

Gibbons show how to perform these tests using a Maximum Likelihood calculation.

6.2 Multivariate test of the CAPM - GRS

However, ideally want to use a test statistic to answer only one question, whether the market portfolio m mean variance efficient.

If we use test on individual securities, we run a regression

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}$$

Then, by the CAPM, MV efficiency implies that

$$\alpha_i = r_{zt}(1 - \beta_i)$$

for all securities i.

One way to test MV efficiency would then be to test $\alpha_i - r_{zt}(1 - \beta_i) = 0$ for all the securities in the sample at (say) the 5% level. The problem is then to aggregate this. Even if the null is true, we expect to reject it in 5% of the cases. Seeing if $\alpha_i = r_{zt}(1 - \beta_i)$ for all i is of course one possibility, but this is extremely conservative. If we wanted to be less conservative, how many rejections of the null for individual securities would we need to reject it for the market?

Hence, we are interested in aggregating over all assets in testing whether the market portfolio m is M-V efficient. H&L 10.17, 10.34-10.40 and MacKinlay and Richardson (1991) (MR) covers this material.

How to test for aggregate MV efficiency:

Consider the estimation of the two following models:

Unconstrained model

$$r_{jt} = \alpha_j + \beta_j r_{mt} + e_{jt}$$

Constrained model

$$r_{jt} = r_{zt}(1 - \beta_j) + \beta_j r_{mt} + e_{jt}$$

The constrained model is a special case of the unconstrained model.

If the CAPM is true, and m *is* MV efficient, the constrained model is the true model. Hence, our estimate of α_j in the unconstrained model should be approximately equal to $r_{zt}(1 - \beta_j)$ (the intercept in the constrained model)

All the multivariate tests of MV efficiency does is to compare the fit of these two models. If the difference is large (according to some statistical metric), reject MV efficiency. Otherwise accept it.

The difference between the methods lies in how to measure the (statistical) difference in fit of the two models. We discuss two methods. The first is covered in H&L10.34-10.40. The original article is Gibbons (1982). These test statistics relies on using Maximum Likelihood to do the estimation. We make the distributional assumption that all errors are multivariate normal. Define:

$$r_t = \begin{bmatrix} r_{1t} \\ \vdots \\ r_{nt} \end{bmatrix} \quad \alpha_t = \begin{bmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{nt} \end{bmatrix} \quad \beta_t = \begin{bmatrix} \beta_{1t} \\ \vdots \\ \beta_{nt} \end{bmatrix} \quad \text{and } e_t = \begin{bmatrix} e_{1t} \\ \vdots \\ e_{nt} \end{bmatrix}$$

The model is then written as

$$r_t = \alpha_t + \beta_t r_{mt} + e_t$$

with the distributional assumption

$$e_t \sim N(\mathbf{0}, V_t)$$

where V_t is the covariance matrix $E[e_t e_t'] = V_t$.

We find the estimates by maximising the log-likelihood ℓ_T with respect to the parameters of interest.

$$\ell_T = -\left(\frac{NT}{2}\right)\ln(2\pi) - \frac{T}{2}\ln\left|\widehat{V}_e\right| - \frac{1}{2}\sum_{t=1}^{T}\hat{e}_t'\widehat{V}_e^{-1}\hat{e}_t$$

We calculate the same function, but now using the estimates $\widehat{V}_e^c$ from the restricted model

$$\ell_T^c = -\left(\frac{NT}{2}\right)\ln(2\pi) - \frac{T}{2}\ln\left|\widehat{V}_e^c\right| - \frac{1}{2}\sum_{t=1}^{T}e_t^{c\prime}(\widehat{V}_e^c)^{-1}e_t^c$$

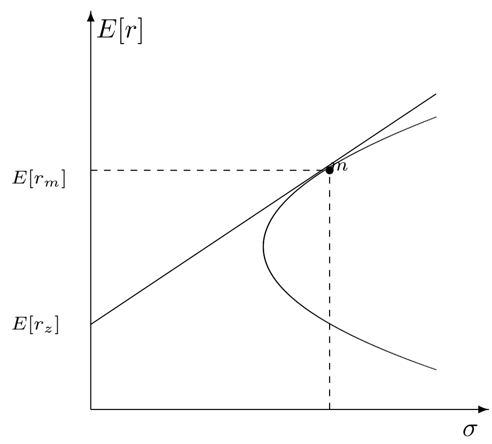
The test statistic we use to test whether m is MV efficient is then

$$-2(\ell_T^c - \ell_T) = T(\ln|\widehat{V}_e^c| - ln|\widehat{V}_e|)$$

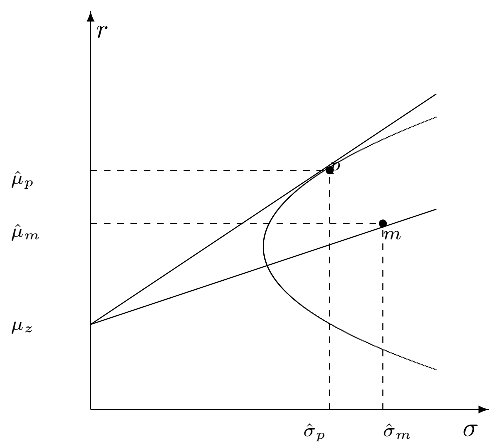
It can be shown that this converges to a χ^2 distribution, and we use this to make probability statements about the outcome.

Let us look at some geometric intuition:

We are interested in a portfolio m. What we would like to know is whether m was on the MV frontier in the ex ante case:



In *ex post* MV space, we can always form the *ex post* efficient frontier:



Here m is the ex post outcome for the portfolio m and p is an ex post frontier portfolio. Intuitively, the test statistic measures the difference in the slope of the two lines in the picture. If this difference is large, we think that the market portfolio is not ex ante efficient.

7 Multivariate test in a GMM setting.

This statistic was developed under distributional assumptions that allowed us to use ML. What if these are not fulfilled, can we still construct a similar test statistic? This is done in the paper of MacKinlay and Richardson (1991) (MR). They construct a test statistic that essentially tests the same restriction, that $\alpha_i = E[r_{zt}](1-\beta_i)$, but in a GMM framework, not a ML.

The setup is as follows.

Again, we have the usual regression

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}$$

We assume that

$$E[\varepsilon_{it}|r_{mt}] = 0$$

This implies two moment restrictions for each asset i:

$$E[\varepsilon_{it}] = E[(r_{it} - \alpha_i - \beta_i r_{mt})] = 0$$

$$E[\varepsilon_{it} r_{mt}] = E[(r_{it} - \alpha_i - \beta_i r_{mt}) r_{mt}] = 0$$

The model is exactly identified. We can "stack" these moment conditions and estimate the parameters $\{\alpha_i, \beta_i\}$ of the model, by the usual formulation using sample moments.

The tests discussed in the paper are different ways of testing the parametric restriction $\alpha_i = 0$. Although they are implemented differently, they all give the same asymptotic result, only they have different short-sample properties. There are three main types of hypothesis tests (See Newey and West (1987))

- Wald-type: Estimate unconstrained model, test parameter restriction.
- Lagrange-Multiplier type: Estimate constrained model, test fit of the (constrained) model.
- Likelihood Ratio type: Compare fit of the unconstrained and constrained model.

The choice of one of these will depend on which of the constrained or unconstrained model is easiest to estimate. As a rule, the third type test tend to perform best, and is preferred if it can be calculated.

The tests shown in MR are cases of these generic types.

The specific cases they look at (student t, normal) they find by explicitly solving the general GMM formulas in cases where the distributions are known. The algebra is a bit tedious, but they are just special cases of the general GMM formulations.

8 Anomalies

Another challenge to the CAPM testing has come from the literatur on *anomalies.* An anomaly is some observable characteristic of an asset that is *not* its beta and is useful in explaining asset returns.

The most famous anomalies is

- Firm size (Banz (1981)) and
- January.

9 The Roll critique

9.1 What is the problem?

The question we have let lie up to now goes under the name of the Roll critique (Roll (1977)). This is concerned with the unobservability of the market portfolio, and its consequences for empirical testing.

If we interpret the CAPM as an equilibrium model, the portfolio m is the return on all assets in the economy, not only the stock market indices we usually use as the market portfolio. Hence, rejecting/not rejecting the models in the tests above may not be viewed strictly as tests of the CAPM.

One solution of this is to reinterpret the one-index formulation as tests of a single factor APT.

The usual way of addressing the problem is to assume that the stock market portfolio m is a sufficient statistic for the market. Let $\hat{m}$ be the market proxy and m the true, unobservable market. If $\rho(r_{\hat{m}}, r_m) = 1$, the beta estimates using m will be equal to the ones we would have gotten using m. See H&L 10.12.

9.2 The Stambaugh (1982) paper

So the assumption made is that the stock market index is "close" to the true market portfolio in the sense of having a unit beta relative to it. One paper that tries to look at how "close" the stock index is to the "true" market empirically is Stambaugh (1982). He adds a large number of assets in addition to the stock market portfolio (bonds, real estate, household investment etc), and looks at the sensitivity of the conclusions of the empirical tests to which portfolio is chosen. He finds that the conclusions are not sensitive to the assets in the portfolio, which makes us more confident about using only the stock market data. We may believe that the stock market index is a *sufficient statistic* for the whole market, or at least that it captures a good deal of the variability of the market as a whole.

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