Are There Tax Effects in the Relative Pricing of U.S. Government Bonds?*

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Abstract

We investigate the impact of the Tax Reform Act of 1986 on the relative pricing of U.S. Treasury bonds. We obtain positive, statistically and economically significant estimates for the implicit tax rates of a "representative" investor in the late 1970's and early 1980's. After the 1986 Tax Reform the point estimates for the tax rate are close to zero. Tests for a regime shift associated with the 1986 Tax Reform support the hypothesis that this event largely eliminated tax effects from the term structure. We discuss both institutional and statutory explanations for this change.

1

Whether there are tax effects in the relative pricing of government bonds is important for both practical and theoretical reasons. Tax effects are informative about tax incidence and the role of different market participants in influencing prices. Practitioners also have a natural interest in the question. If relative prices reflect the tax status of one group of investors, others can exploit these differences in their portfolio decisions. In the government bond market, individual investors are taxed very differently than dealers and many institutions. Pricing that reflects the tax positions of individual investors creates profitable trading opportunities for dealers, and pricing that does not reflect the tax status of individual investors gives them motives to trade.

This papers tests for the presence of tax effects in prices of U.S. Treasury bonds, and evaluates changes in these tax effects around the tax reform in 1986, which reduced the asymmetries between taxation of market discount and premium bonds. We begin by using parameterized, flexible functional forms to represent the after-tax term structure, and estimate, at each date, both the tax rate implicit in relative prices and the parameters of this functional form. These estimated tax rates drop markedly in the mid-1980's. Evaluating the significance of this change requires a structural model of the after-tax term structure, which holds fixed through time the term structure parameters. Using such a model we find convincing evidence for a regime shift around 1986.

Researchers have used a number of methods to document the presence of tax effects in the relative prices of government bonds. Schaefer [1982], Ronn [1987], and Dermody and Rockafellar [1991] use linear programming to identify optimal portfolios for investors with specific tax rates. Jordan [1984] and Litzenberger and Rolfo [1984a] use non-linear regression methods to estimate the discount factors, or term structure, for a given tax rate. Evidence of a tax effect is then sought in differences between the criterion functions associated with different tax rates. Finally, researchers such as Litzenberger and Rolfo [1984b], Jordan and Jordan [1991], and Kamara [1994] have focused on particular subsets of bonds that are identical except for their tax status, such as "triplets" with identical maturities, or notes and bills with the same maturity.

All of these approaches have produced evidence of "tax effects" in the relative pricing of bonds, but the persistence of these effects through time, and their statistical and economic importance remain unresolved issues. We use a sample that includes data from both before and after 1986, so we can evaluate its importance for tax effects in bond prices. Also, in contrast to previous studies, our non-linear regression approach provides standard errors for the term structure and tax rates estimated simultaneously. This allows us to ask whether the behaviors interpreted as tax effects can be reliably attributed to something other than chance—whether the behaviors observed are persistent, systematic, and pervasive enough to appear large in a statistical metric. Finally, past studies have approached the estimation problem on a period-by-period basis. This only uses the information in the cross section of bond prices at a given date to estimate the parameters for that date, and ignores the possibility of correlation in the estimates between dates. Since both term-structure parameters and tax rates are allowed vary through time, there is no natural test for a

regime shift at a point in time. To provide such a test, we also estimate a multiperiod model that holds deep parameters fixed and exploits information in the time series. In this way we can ask whether important events, such as changes in the tax law, have altered the structure of relative prices in a significant way.

Like earlier researchers we find, in the late 1970's and early 1980's, estimates of the tax rate that are consistently positive. They appear statistically different from zero. At the time of the 1986 Tax Reform, however, the point estimates of the tax rates drop dramatically, and thereafter are close to zero. While we report results using the functional form suggested by the Cox et al. [1985] (hereafter, CIR) model, this conclusion appears to be robust to the method used to fit the term structure.

The drop in the estimated tax rates suggests a regime shift associated with the 1986 Tax Reform. Estimating the CIR model as a multiperiod model for the after-tax term structure, we find strong evidence of such a change. Prior to 1986 the estimated tax rate is positive and significantly different from zero. After 1986, the point estimates are very close to zero. Quasi-likelihood ratio tests reject the null hypothesis that the estimated tax rate does not change across this point in time. This conclusion also appears robust to the particular specification employed.

We emphasize that our focus is limited to the *relative* pricing of government bonds. Their pricing relative to other categories of securities is not at issue. For instance, there are very obvious tax effects in the pricing of government or corporate bonds relative to municipal securities, but arbitrage across these categories of bonds for dealers and tax-exempt institutions is prohibited by tax law. These groups can trade freely to exploit any tax effects in evidence in the relative prices of taxable bonds.

The remainder of the paper is organized as follows. The next section discusses in more detail the hypotheses we test. Section 2 then describes the data. Section sec:month-by-month details the estimation strategy and presents results using a period-by-period approach, similar to past work. Section 4 does the same for the multiperiod estimation, which holds the structural parameters of the after-tax term structure model fixed through time and tests for a regime shift. We conclude in Section 5 with a summary of our results and discussion of reasons for the diminished tax effects after 1986. An appendix details the modeling of the after-tax cash flows, accounting for such complications as off-cycle first coupons, accrued interest, and premium amortization.

1   Hypotheses and Model

The null hypothesis we wish to evaluate is a simple one:

Hypothesis 1 (No Tax Effects) *All bonds are priced so that they could optimally be held by some investor with a tax rate of zero.*

Both practical and theoretical arguments can be advanced in support of this hypothesis. Tax-induced price differentials must be attributable to differences in the treatment of interest income, return of principal, and capital gains or losses. Dealers, however, are taxed symmetrically on interest income and capital gains, and many large financial institutions are not taxed at all. Thus, if tax-induced price differentials are present, they create a motive for these agents to trade. If these parties do trade to eliminate any such tax effects, this will create motives for individual investors, for whom the distinction between ordinary income and capital gains is relevant, to trade in the opposite direction. Which group ultimately determines prices is an empirical issue. There are at least two reasons, however, to expect dealers and institutions to dominate in this process.[1]

1. In exploiting asymmetries between the way they are taxed and the relative pricing of bonds all parties face transaction costs. Dealers and large tax-exempt institutions presumably trade at lower costs than individual investors.

2. Strategies by which individuals would exploit their tax position typically involve transforming ordinary income into capital gains. Doing this requires generating interest expense or losses elsewhere in the portfolio to shield income from taxation. That is, tax-arbitrage strategies involve shorting one security, or borrowing, and using the resulting deductions or losses to offset income from a long position. The tax code, however, limits investors' investment interest expense to investment income, and similarly limits the use of capital losses. The profitability of dealer arbitrage activities is not limited by the tax code.

The alternative hypothesis we investigate is that the pricing of the bonds reflects the relative valuations of an investor with a positive tax rate. Let $d_t(t_m)$ be the price at time t of an after-tax claim of one dollar to be delivered at time t_m.[2] Then a default free coupon bond that pays 100 at maturity, and is currently at a market discount, will be priced as follows:

$$\hat{P}_{jt} = C_j(1 - \tau_{it}) \sum_{m=1}^{M_j} d_t(t_m) + 100 d_t(t_{M_j}) - (100 - \hat{P}_{jt})\tau_{gt} d_t(t_{M_j}), \tag{1}$$

where $\hat{P}_{jt}$ is the price of bond j at time t, M_j is the number of coupons remaining for bond j, and t_m is the time of payment of coupon m. The ordinary income and capital gains tax rates are τ_i and τ_g, respectively. This price represents the marginal valuation of a bond at a market

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[1]Green [1993] relies on such arguments in his attempt to explain the anomalous behavior of the taxable and tax-exempt term structures. In that paper, portfolios of taxables are priced relative to tax-exempts to reflect their tax advantages. The prices of all taxable bonds then inherit these characteristics because of the assumption that dealer arbitrage eliminates any tax effects in the way taxables are priced relative to each other. Our results, which demonstrate tax effects are present in the relative pricing of taxables before 1986, are inconsistent with that explanation, although not with the empirical findings in Green [1993], which reject the model in the pre-1986 period and support it after the TRA.

[2]In an equilibrium model this would be the agent's expected marginal rate of substitution for nominal cash flows.

discount to a taxable investor. The formula ignores the effects of tax options on the agent's implicit valuation of the bond.[3] We are assuming that the representative investor contemplates holding the bonds to maturity, which is consistent with the existing literature estimating implied tax rates. Less significantly, expression (1) oversimplifies by ignoring the treatment of accrued interest, the asymmetric size of the first coupon, the amortization of the discount after 1984, and the asymmetric treatment of premium and discount bonds. For premium bonds the difference between the market price and face value can be amortized against coupon income over the remaining life of the bond. For bonds at a market discount issued after 1984, the discount must be amortized and the amortized portion of the proceeds from sale or maturity are treated as ordinary income when sale or maturity occur. In our estimations, we account for these aspects of the tax code. The Appendix provides a complete description of the adjustments made to the basic present value relation (1).

Throughout our analysis, we constrain the relationship between tax rates for capital gains and ordinary income to be as specified by the tax statutes. For capital gains of less than one year, $\tau_g = \tau_i$. For long-term gains, we set $\tau_g = 0.4\tau_i$ from 1978–86, while from 1987–92, $\tau_g = \min(\tau_i, 0.28)$. As is detailed in the Appendix, for bonds issued after the 1984 tax bill a market discount must be amortized as ordinary income, but the resulting income is only realized on sale or maturity. Thus, for bonds held to maturity, the market discount is taxed, at maturity, as ordinary income, and we cannot separately estimate a capital gains rate. The distinction between these new bonds and the older bonds becomes moot after 1986. At that point, as long as the investor's ordinary rate is less than 28%, which it is in our estimates, the only difference between income and capital gains is in the timing of realization, so the pre-1984 and post-1984 bonds are taxed in the same way.

Solving the pricing equation (1) for $\hat{P}_{jt}$ gives:

$$\hat{P}_{jt}(\tau_t, d_t(\cdot)) = \frac{C_j(1 - \tau_{it}) \sum_{m=1}^{M_j} d_t(t_m) + 100(1 - \tau_{gt})d_t(t_{M_j})}{1 - \tau_{gt}d_t(t_{M_j})}, \tag{2}$$

where $\tau_t = \{\tau_{it}, \tau_{gt}\}$, the ordinary and capital gains tax rates, respectively. This pricing relationship is the basis for the estimation done in the paper.[4] We assume that that (2) gives an unbiased estimate of the true price, under either the null ($\tau_t = 0$) or an alternative that posits ($\tau_t > 0$). Lacking a model of any differences between observed prices and the theoretical price, $\hat{P}_{jt}$, the choice of moment conditions used to estimate tax rates and test the fit of this pricing formula is somewhat *ad hoc*. We assume that at the true parameters the observed price, P_{jt}, is equal to the model price,

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[3]A number of papers point to the option value of timing the realizations of capital gains and losses, which will influence the price. Examples include Constantinides (1984) and Constantinides and Ingersoll (1984). See Litzenberger (1989) for an overview.

[4]This differs from the specification used by McCulloch (1975), Jordan (1984), and Litzenberger and Rolfo (1984a). They express (1) in a form that is linear in the parameters of the cubic splines they employ as an approximating functional form, but which leaves the price on both sides of the equation. They then apply instrumental variables procedures to account for the dependence between the residual and the independent variable. Since we do a fully nonlinear simultaneous estimation of the term structure and the tax rates, this is unnecessary.

$\hat{P}_{jt}(\tau_t, d_t)$, plus an error term

$$P_{jt} = \hat{P}_{jt}(\tau_t, d_t) + \varepsilon_{jt}. \tag{3}$$

where the errors have zero mean and norms minimized at the "true" parameter values.

The estimation problem is then the following: from observed bond prices, both in cross section and time series, we want to estimate the tax rates τ_{it} and τ_{gt}. This can not be done directly, since the after-tax discount factors, $d_t(\cdot)$, are unknown. We therefore estimate the discount function simultaneously with the tax parameters. This requires that we parameterize after-tax the term structure in some lower dimensional way, because there are not enough linearly independent bonds available to estimate all the discount factors. Although we find that our conclusions are quite robust to different ways of parameterizing the term structure, this means that our tests are all joint tests.

The traditional nonlinear regression approach estimates different term-structure parameters and tax rates for each period using only the information in the cross section of bond prices. When one pools the cross-sectional and time-series data, therefore, this approach estimates a model in which the number of parameters grows to infinity as the number of dates in the time series increases. Alternatively, we can employ a structural model that expresses the term structure as a function of a limited set of state variables. If the structural model is correct, we can obtain more precise estimates of a smaller number of parameters, and recognize in our inferences that the estimates of the tax parameters are not independent across time due to autocorrelation.

For both approaches we use the single state variable model of Cox, Ingersoll and Ross (1985), interpreted as a model of the nominal, after-tax term structure. The CIR model offers parsimony, arbitrage-free prices, parameters that have economic interpretations, and a reasonable degree of flexibility as an approximating functional form in cross-sectional estimation. Our conclusions regarding the behavior of the estimated tax rates do not appear to depend on our choice of the CIR functional form. In our cross sectional tests, we also estimate the term structure using the Nelson and Siegel (1987) exponential functional form and cubic splines. As detailed in Section III, this leads to similar behaviors for the estimated tax rates over time.

In the CIR model the equilibrium price of a zero coupon bond maturing at s, which we interpret as the after-tax discount factor, is

$$d_t(s) = A(s)e^{-B(s)r(t)} \tag{4}$$

where $A(s)$ and $B(s)$ are functions of maturity, s, and the parameters of the model. As pointed out in Brown and Dybvig (1986), all of the structural parameters are not separately identified in one cross-section. By focusing only on those parameters that are identified we lose the ability to fully characterize the dynamics of the spot rate, but this is not costly to the extent that our primary focus is on the tax effects embedded in bond prices. Hence, the after-tax term structure is

parameterized by the after-tax spot rate of interest, which can either be identified as a parameter from each cross-section or can be treated as data, and the structural parameters $\{\phi_1, \phi_2, \phi_3\}$, which then give the after-tax discount function as

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\begin{aligned}
A_t(s) &= \left[\frac{\phi_1 e^{\phi_2 s}}{\phi_2(e^{\phi_1 s} - 1) + \phi_1}\right]^{\phi_3}, \\
B_t(s) &= \frac{e^{\phi_1 s} - 1}{\phi_2(e^{\phi_1 s} - 1) + \phi_1}.
\end{aligned}
\tag{5}
$$

In terms of the parameters of the CIR process for the interest rate,

$$
\begin{aligned}
\phi_1 &= \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}, \\
\phi_2 &= (\kappa + \lambda + \phi_1)/2, \\
\phi_3 &= 2\kappa\theta/\sigma^2,
\end{aligned}
\tag{6}
$$

where λ is the "market price of risk," σ is the volatility parameter, θ is the long-term mean, and κ governs the speed of mean reversion. The parameters ϕ_1 and ϕ_2 must be positive, and we impose these constraints in our estimations.[5]

For some of our tests these parameters will be allowed to vary through time. For others they will be held fixed. We start by interpreting the model simply as providing a flexible functional form to approximate the after-tax term structure, and estimate tax rates separately in each period. The behaviors of the estimated tax rates are robust to the choice of the CIR model as a flexible functional form. They also hold using the cubic splines employed by Litzenberger and Rolfo (1984a), or the exponential functions of Nelson and Siegel (1987).

We also estimate the model holding the parameters in (5) fixed through time, pooling the cross sectional and time series data. In this way we can address with formal tests such questions as whether the tax effects in pricing disappear after the 1986 Tax Reform, as suggested by our period-by-period estimations.

Under either the month-by-month, or the multiperiod approach, we will interpret positive estimated tax rates as evidence of tax effects in the relative pricing of tresury bonds.

2 Data

We use data on U.S. Treasury bonds provided by the Center for Research in Securities Prices (CRSP). We exclude callable and flower bonds, and bonds for which we do not have both bid and ask prices. Our tests cover the period 1978 to 1992. Before 1978 the number of long-term bonds

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[5]Under the constraint $\phi_3 > 1$, zero becomes inaccessible for the interest rate process in the CIR model. Imposing this constraint leads to estimates that are qualitatively similar to the results reported here. Such estimates are reported in earlier versions of the paper, available from the authors on request.

is small, which makes it difficult to estimate the term structure. Our data cover portions of the sample periods studied in earlier papers, such as Litzenberger and Rolfo [1984a], allowing us to compare our results with earlier work.

Table 1 Table I: Premium-discount distribution.

	Average	Price					
Year	No. of Bonds	<90	90–95	95–100	100-105	105–110	110<
1978	122	1.37	7.99	59.8	30.4	0.273	0
1979	127	4.64	14.2	58.6	22.2	0.327	0
1980	132	15.9	19.3	42	18.3	2.39	1.95
1981	139	23.5	18.8	35.1	18.1	3.76	0.597
1982	148	12.9	10.9	28	22.9	14.5	10.8
1983	158	3.94	8.72	27.7	24.2	15.9	19.4
1984	168	7.86	11.5	29.4	27.3	15.2	8.7
1985	179	1.86	4.41	20.5	25.8	25.1	22.4
1986	190	0	1.35	17	19.4	18.6	43.6
1987	195	0.81	3.67	24.4	24.9	15	31.2
1988	200	0.958	4.16	27.9	31.3	12.6	23.1
1989	204	0.408	4.57	26.6	33.5	13.9	21
1990	208	0.68	2.8	23.5	42.3	14.3	16.4
1991	215	0.116	1.12	16.4	33.4	27.2	21.7
1992	221	0	0.15	17.2	24.5	26.4	31.8

The table lists percentages, averaged across months within each year, of the samples with prices in the specified intervals for each year. Flower bonds and callable bonds have been excluded from the sample.

Table 1 describes the distribution of bond prices. The table lists percentages, averaged across months within each year, of bonds with prices in various intervals around par. Many of the tax effects attributed to default free bonds are due to the differential treatment of income and capital gains for bonds at market discounts versus premiums. To distinguish these effects requires that the cross section includes a range of bonds priced at discounts and premiums. As is evident from the table, there are several periods when very few discount bonds are available. This also speaks to the advantages of estimating a structural model, with constant parameters through time, where the information in one cross section is used to estimate parameters for the term structure in other periods.

Table 2 shows average percentages of bonds in the listed maturity intervals. Having a broad cross section of maturities is important for several reasons. Tax effects in relative pricing imply differences in the valuation of coupon payments, which are fully taxed as ordinary income, and principal payments, which for a par or premium bond are tax-free. If a broad range of maturities is not available, it becomes more difficult to estimate separately the value of each type of payment at a given date. Another important possible cause of tax effects in bond pricing is the opportunity market discount bonds offer to defer taxation. The value of deferral depends on the maturity of the

Table 2 Table II: Distribution of Maturities.

Year	Maturity in Years								
	0-1	1-2	2-3	3-5	5-7.5	7.5-10	10-12.5	12.5-15	15-20	20-30
1978	23.0	25.0	14.1	20.1	6.6	5.6	1.2	4.1	0.4	0.0
1979	23.0	25.7	15.7	14.4	7.1	5.3	1.1	7.3	0.3	0.0
1980	23.3	26.1	11.9	14.4	7.6	5.6	2.1	8.8	0.3	0.0
1981	23.3	21.9	11.0	15.1	9.3	6.0	4.7	6.5	2.0	0.3
1982	19.9	19.8	12.4	15.5	10.6	7.2	6.8	3.2	4.4	0.2
1983	17.9	19.8	11.6	16.3	11.5	9.4	6.5	0.3	6.5	0.3
1984	18.1	18.9	11.6	16.5	11.7	10.7	4.0	0.0	8.2	0.2
1985	17.3	18.8	11.1	16.6	12.5	11.9	1.1	0.0	9.6	1.1
1986	17.4	17.4	11.8	16.3	13.9	10.1	0.0	1.4	8.7	3.1
1987	16.7	18.4	11.7	16.6	15.3	7.2	0.0	3.1	6.7	4.3
1988	17.6	18.1	10.9	18.1	15.4	5.0	0.3	4.5	4.7	5.3
1989	17.5	17.6	11.3	19.0	13.2	5.4	2.3	4.2	2.8	6.6
1990	16.9	17.5	13.1	19.4	10.4	5.7	3.5	4.4	1.1	8.0
1991	28.9	15.9	10.6	16.6	7.9	5.6	3.5	2.9	0.0	8.0
1992	29.5	15.0	9.9	16.5	8.1	6.3	3.5	1.5	0	8.8

For each year of the sample, the table lists the average percentage of the sample with maturities within the given interval.

bond. Finally, absent a wide range of long-maturity bonds, it becomes difficult to estimate the term structure with any confidence. The resulting uncertainty concerning the term-structure parameters will increase the standard errors of the tax rates and reduce the power of our test statistics.

3 Month-By-Month Estimations

We begin by estimating all the parameters of the model period-by-period. These results provide a useful starting point because they can be compared directly with the nonlinear regression results in such previous research as McCulloch [1975], Jordan [1984], and Litzenberger and Rolfo [1984a].

3.1 Estimation Strategy

Assuming that the after-tax term structure is given by the CIR formula, (5), our task is to estimate the tax rates $\{\tau_{it}, \tau_{gt}\}$, and the parameters of the term structure function $\{\phi_{1t}, \phi_{2t}, \phi_{3t}\}$. For a given t, these parameters can be identified from the cross section of bond prices. We treat the spot rate in the CIR formula as data, by taking the one-month T-bill rates from the data base created by Fama and provided by CRSP, and multiplying these by $(1 - \tau_t)$, where the tax rate is one of the parameters to be estimated. As Pearson and Sun [1994] point out, a discretely sampled short-term rate is an imperfect proxy for the state variable in the CIR model. We show below, however, that very similar results obtain when we estimate the state-variable from the cross section of bond prices. Brown and Dybvig [1986] report point estimates from similar regressions without

accounting for taxes, and treating the short rate as a parameter to be estimated from the cross section.

Let γ_t represent the parameters to be estimated. Our maintained assumption is that the true parameters, γ_{0t}, uniquely minimize the squared differences between the observed and theoretical price, which is given by (2):

$$\text{SSE}_t(\gamma_t) = \sum_{j=1}^{J_t} E\left[(P_{jt} - \hat{P}_{jt}(\gamma_t))^2\right]. \tag{7}$$

Here j indexes the bond in cross section. At time t there are J_t bonds available.

Alternatively, we can express this assumption in terms of the first-order conditions for this minimization. That is, the true parameters, $\{\gamma_{0t}\}_t$, must solve:

$$E\left[\frac{\partial \hat{P}_{jt}(\gamma_{0t})}{\partial \gamma_t}\left(P_{jt} - \hat{P}_{jt}(\gamma_{0t})\right)\right] = 0. \tag{8}$$

Thus, we can view the gradients of the prices with respect to the parameters as instrumental variables in a GMM estimation, and calculate point estimates by solving the sample equivalent of these moment conditions. Estimating the variance-covariance matrix of the orthogonality conditions, gives estimates of standard errors for the estimated tax rates, which are consistent with heteroskedastic and correlated errors within the cross section.

3.2 Results

In Figure 1 we plot the point estimates of the tax rate, and two GMM standard errors in either direction. These estimates display certain broad features that are very robust to the specification we employ.
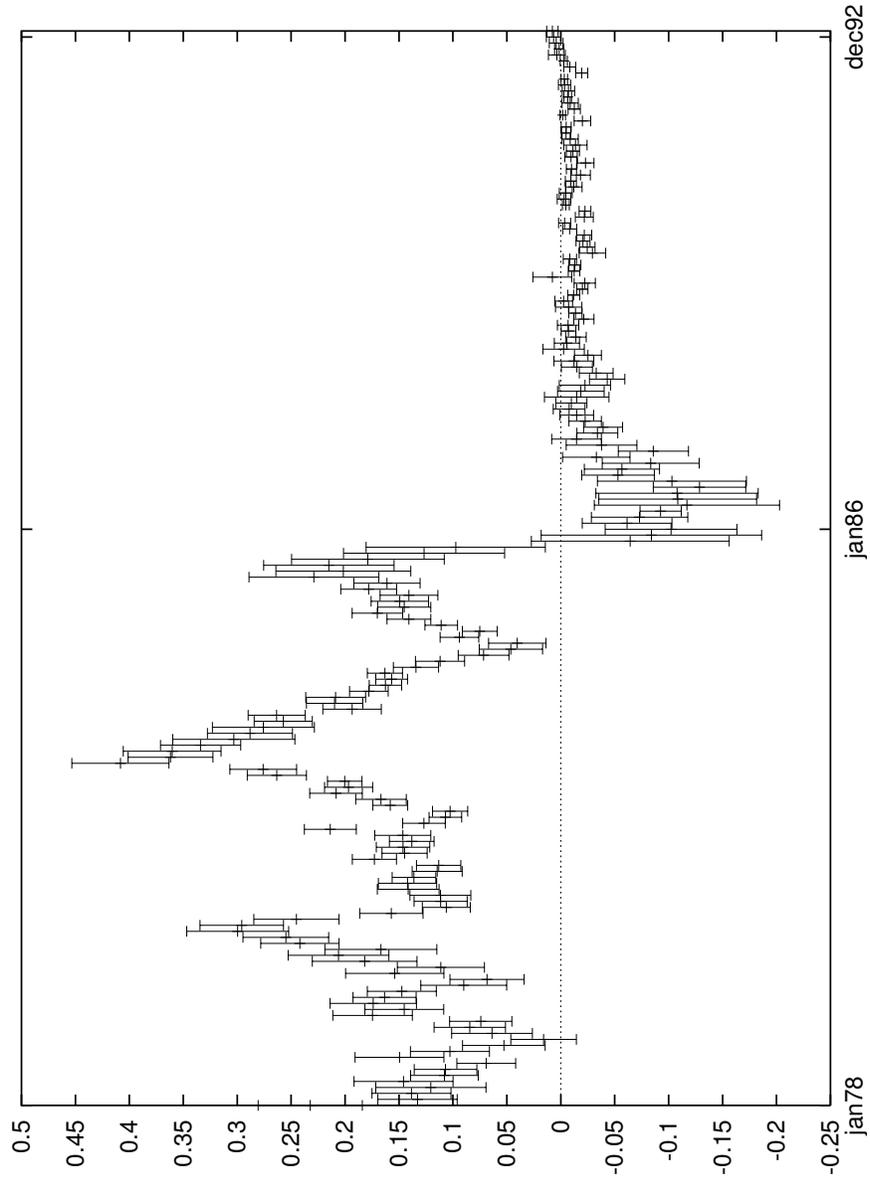
First, while the point estimates of the tax rates are consistently positive prior to 1986, they appear quite volatile in time-series, and highly autocorrelated.

Second, the average or typical values seem unreasonably low relative to the relevant statutory rates. These were well above the 10%–15% levels often obtained in the estimation in the early years of the sample period. As we discuss in more detail below, this could be due to clientele effects in the pricing of the bonds, or to the omission of the timing options for gains and losses in our specification. The low average point estimates could also be due to the difficulty of separately identifying the structural and tax parameters using only cross sectional information.

Finally, the point estimates drop dramatically at the time of the 1986 Tax Reform, and remain economically insignificantly different from zero thereafter.[6]

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[6]This finding is consistent with the results of Beim [1991], who employs a variant of the linear programming method in the period 1987–89, and finds little support for tax clienteles as an explanation of bond pricing.

Figure 1 Estimated income tax rates using monthly CIR term structure from 1978 to 1992

Heavy black dots represent the point estimates of the tax rates. The intervals around the dots include two standard deviations in each direction. The plot illustrates the drop in estimated tax rates around 1986, at the time of the Tax Reform Act.

The conclusions we draw from the period-by-period estimation in Figure 1 are that there may well have been tax effects important to relative pricing in the treasury bond market prior to the 1986 Tax Reform, but that these tax effects dramatically diminish after 1986. One simple test of whether this change is meaningful is to compare the mean estimated tax rates over the two subperiods. We follow Litzenberger and Rolfo [1984a] in treating the estimated tax rates as data. Our test for a regime change is then just a test for equality of means across the two subperiods. The mean tax rate over the first subperiod (96 observations) is 16.05%, and the mean tax rate over the second subperiod (84 observations) is -2.49%. A heteroscedasticity and autocorrelation robust test for a regime shift in 1986 yields a likelihood ratio statistic of 226.41, which would reject the null of equal means across the two periods at any reasonable significance level.[7]

This test suggests the difference in sample means across the two regimes is large, relative to their variance. Such a test, however, ignores the fact that the term structure is estimated rather than known. We would like to account, in our tests, for the covariances between the estimates of the tax rates and all the other estimated parameters, both at a point in time and across time. When the term structure is estimated month-by-month, accounting for the covariances between the term structure parameters and the tax rates at different dates is infeasible, because of the number of parameters grows with the length of the cross section. By holding the term structure model fixed through time, and interpreting the CIR model as truly dynamic, we can perform a test for changes in the tax regime while still recognizing the estimation error in the term structure.

The broad behaviors evident in Figure 1 do not appear to be an artifact of our particular specification. Figures 2 and 3 provide some evidence concerning robustness.

Figure 2 plots the point estimates of the tax rates obtained as described above, treating the spot interest rate as data, along with those obtained treating the spot interest rate as a parameter, as in Brown and Dybvig [1986]. These estimates follow each other closely through time. This leads us to treat the spot rate as data in our multiperiod estimates later, since it dramatically reduces the number of parameters to be estimated.

In Figure 3 we plot the estimated tax rates obtained using three different flexible functional forms for the after-tax term structure: the CIR pricing formula, the exponential form recommended by Nelson and Siegel [1987], and a cubic spline similar to that employed by Litzenberger and Rolfo [1984a].[8] As is evident in the figure, the Nelson-Siegel functional form and the CIR pricing formula behave in very similar ways. The cubic spline leads to much more volatile tax rate estimates, but

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[7]The test we employ is that suggested in Andrews and Fair [1988], calculated using the Newey and West [1987] weighting matrix with 5 lags. The LR-statistic is asymptotically chi-squared with one degree of freedom.
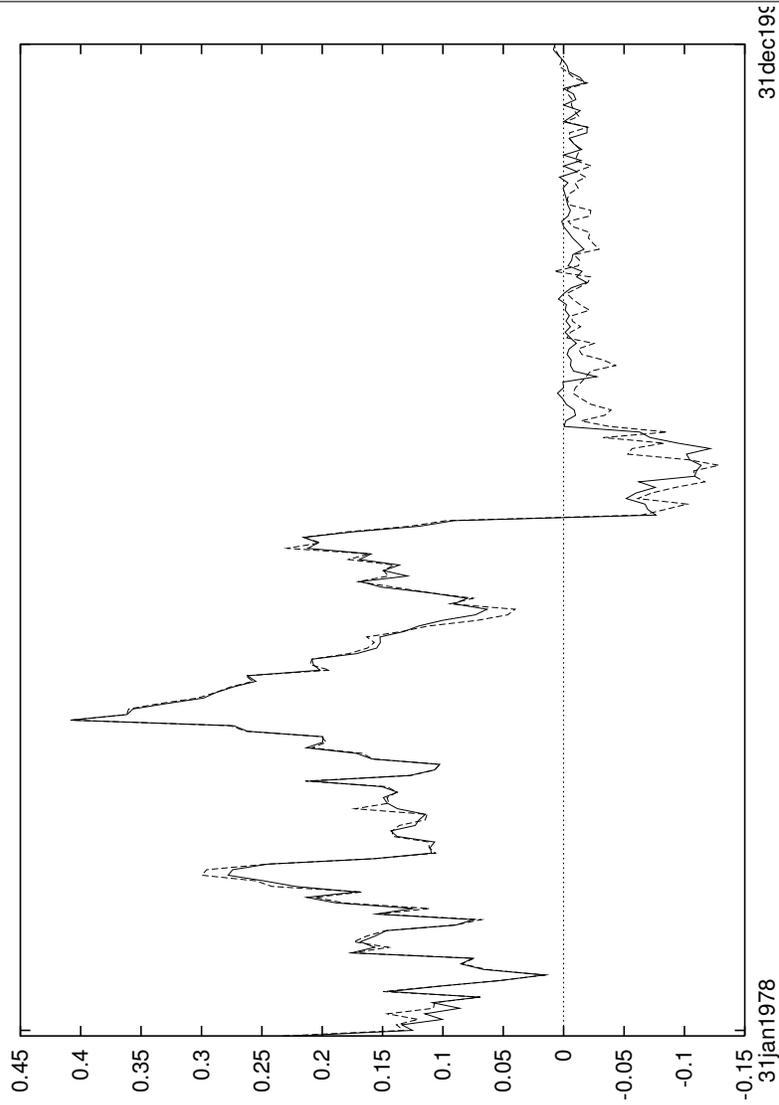
[8]The plot uses two additional functional forms for the term structure. The exponential functional form is

$$r(s) = \beta_0 + (\beta_1 + \beta_2)\left[\frac{1 - e^{-\frac{t}{\lambda}}}{\frac{t}{\lambda}}\right] + \beta_2\left[e^{-\frac{t}{\lambda}}\right]$$

$$d(t) = e^{-r(t)t}.$$

This has 4 parameters to estimate: $\{\beta_0, \beta_1, \beta_2, \lambda\}$. The cubic spline is the same parameterization used by McCulloch

12

Figure 2 Estimated income tax rates using different proxies for the spot rate

The term structure is estimated for every month using the CIR model. The solid line plots the tax rates obtained when estimating the spot rate as a parameter. The dashed line plots the rates obtained using one-month Tbill rates as the (before-tax) spot rate.

13

Figure 3 Estimated income tax rates obtained using different parametric forms for the after-tax
term structure
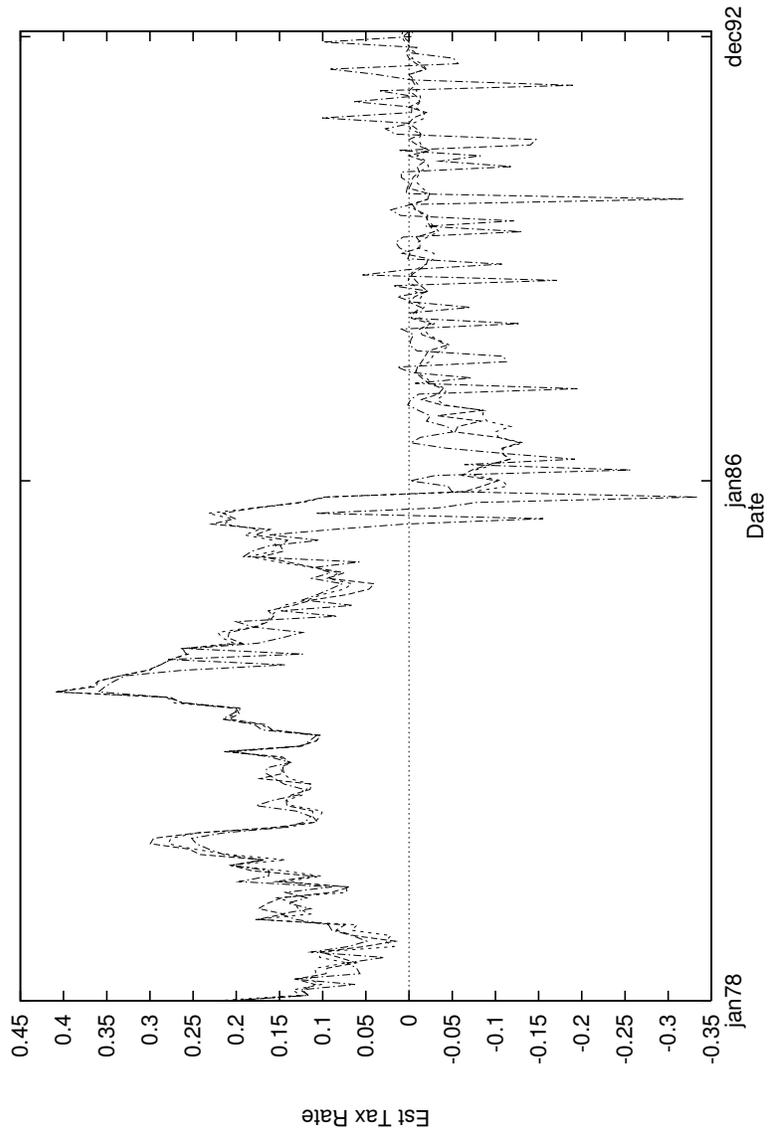
The term structure is estimated month-by-month from 1978–1992 using the CIR pricing formula (solid line), the Nelson and
Siegel [1987] exponential function (dashed line), and a cubic spline (dotted line).

14

these show an overall pattern through time similar to the CIR-based estimates. These results are reassuring about the ability of the CIR pricing formula to flexibly represent the more important behaviors in the after-tax term structure.

Finally, the time-series volatility and the low levels of the point estimates early in the sample period may suggest misspecification of the tax environment. For example, these outcomes may be due to clientele effects. Different subsets of bonds could be held by different tax clienteles, and the tax rate of the representative, marginal investor ends up as a (nonlinear) average of these. Also, the buy and hold strategy assumed rules out the possibility of increasing after-tax returns through dynamic trading strategies. The bonds for which these options are most valuable will appear to have high values relative to the after-tax cash flows postulated by our approach. Constantinides and Ingersoll [1984] report that in their simulations the value of the timing option, as a function of the coupon, has an inverted-U shape. It is higher for bonds near par. Since these are bonds that also are relatively heavily taxed, reconciling their higher values with the model might force the tax rate down.

Table 3 Determinants of Pricing Errors

	constant	coupon	maturity	age	premium	R–sqr
$\tau = 0$	.9114	.1927	.0479	.0362	-.2454	36.1%
	(94.60)	(74.46)	(56.58)	(20.21)	(-19.70)	
τ freely estimated	0.0025	-0.0023	0.0087	0.0184	-.0157	2.8%
	(0.52)	(-16.94)	(19.76)	(19.80)	(-2.43)	

The residuals from the fitted after-tax term structure estimated period-by-period are regressed on bond characteristics associated with tax clienteles. The t-statistics are in parentheses. There are 31,258 observations for each regression.

In fact, though, our estimation procedure appears to be doing a reasonable job of purging the data of tax effects. Table 3 reports the results of OLS regressions of the residuals from our estimation on characteristics of the bonds (coupon, time to maturity, and a dummy variable equal to one if the bond is at a premium) that have traditionally been associated with tax clienteles. We also include the time since the bond was issued (age) as a proxy for the bond's liquidity, since various researchers have shown this characteristic to be important in pricing. (See Warga [1992] and Kamara [1994].) These regressions were run, first, using errors from fitting the CIR term structure period-by-period with the tax rate constrained to be zero. They show behavior similar to that reported by Brown and Dybvig [1986], who argue that the directions in which the CIR model fails to fit seem related to characteristics of the bonds which would suggest tax effects. In our regressions,

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[1971] and Litzenberger and Rolfo [1984a].

$$d(t) = 1 + b_1 t + c_1 t^2 + d_1 t^3 + \sum_{j=1}^{K} F_j (t - t_j)^3 1_{\{t < t_j\}}$$

Here $1_{\{A\}}$ is the indicator function and we have K "knots" in the spline. We set $K = 4$, and so must estimate 7 parameters, $\{b_1, c_1, d_1, F_1, \cdots, F_4\}$.

deviations between prices and theoretical present values are clearly related to coupon levels, and might, thus, be attributable to taxes. In the second panel we report results of the same regressions when the tax rate is freely estimated. Both regressions result in significant t-statistics, but this is hardly a surprise given the huge number of observations. The R^2-statistics are more revealing. The regression using errors when the tax rate is freely estimated has minimal explanatory power, while the R^2 when the tax rate is constrained to zero shows that the term structure approximation systematically misprices certain categories of bonds when taxes are ignored.

4 Multiperiod Term Structure Estimations

A more structural approach offers several advantages over the traditional, period-by-period estimations described in the previous section. The point estimates of the tax rates in the month-by-month estimations are both volatile and autocorrelated. Thus, the fact that they are consistently positive may be misleading. The estimated tax rate in one period is not independent of the others. The multiperiod estimation we do in this section addresses this limitation, by allowing for autocorrelation in the orthogonality conditions across bonds and through time. This would be infeasible without holding parameters fixed through time, as the number of parameters would then grow with the length of the time series. Using this approach we can also pose questions about the significance of the apparent structural shift around the 1986 Tax Reform. The approach taken in the previous section treats each date as a distinct regime, making it meaningless to ask whether the regime changes at a point in time. Finally, requiring consistency through time allows us to use information from all periods to overcome the practical difficulties of separately identifying the tax rates and the term structure.

4.1 Estimation Strategy

Our task is to estimate the CIR parameters, $\{\phi_1, \phi_2, \phi_3\}$, which are fixed through time, and the tax rates, which are fixed across subperiods that represent different regimes. Our approach is to pool the cross sectional and time series data, and assume the true parameters, γ_0, minimize:

$$\text{SSE}(\gamma) = \sum_{t=1}^{T} \sum_{j=1}^{J_t} E\left[(P_{jt} - \hat{P}_{jt}(\gamma))^2\right] \tag{9}$$

Again, the parameters must solve the sample first-order conditions as in equation (8) with $\gamma_{0t} = \gamma_0$. In computing the sample estimates of these orthogonality conditions, however, we must now average across periods as well as bonds, as the choice of the CIR parameters determines the theoretical price of all the bonds in all periods.

The only aspect of the problem that is nonstandard is the estimation of the weighting matrix. We adapt the usual Newey–West procedure by matching maturities across months, so that we

account for autocorrelation both within each cross-section and across time. The need to pair bonds with similar maturities is created by the unequal sizes of the cross sections through time.[9]

This procedure exploits the ordering maturity imposes on the bonds within a period. It seems reasonable that whatever causes observed prices to deviate from intrinsic values might affect bonds in the same portion of the term structure similarly, and we would like to account for this in our estimation. At the same time, if adding bonds to the cross section is to contribute independent information we must assume the dependence decreases as the difference in maturities increases.

The cost of moving to a structural model is, of course, that we lose flexibility in capturing movements in the term structure, and this may lead us, if the model is mispecified, to ascribe to taxes mispricing that a parametrically richer model would attribute to the after-tax term structure. The empirical deficiencies of the CIR model have been extensively documented. For example, Pearson and Sun [1994] report that parameter estimates of the two-factor CIR model are not stable across subperiods. A structural change in the after-tax term structure could lead us to a mistaken inference that the tax effects have changed. We do provide some diagnostics below, however, suggesting this is not the source of our findings.

In addition, in the earlier years of the sample, there is no reason to believe that the marginal tax rate of the representative investor would be constant, and by forcing it to be so we could be understating the importance of taxes in relative pricing. Judging by our estimates, this does not appear to be a severe problem. The tax effects in the pre-1986 period appear very significant under

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[9]The Newey and West [1987] estimate of a covariance matrix of $\{e_t\}$ using L lags, is:

$$S = \sum_{l=-L}^{L} w(l, L)\Omega_l$$

where

$$\Omega_j = E\left[e_t e'_{t+j}\right]$$

and

$$w(l, L) = 1 - \frac{l}{L+1}$$

The data in our estimation mixes cross sections and time series. Let $\{e_{jt}\}$ be the data, where j indexes the cross section and t the time. If the error e_{jt} "matched" the error $e_{j,t-1}$, we could modify the Newey–West procedure to take into account these data in the following way:

$$\hat{S} = \sum_{k=-K}^{K} \sum_{l=-L}^{L} w(k, K)w(l, L)\Omega_{k,l}$$

where

$$\Omega_{k,l} = E\left[e_{jt} e'_{j+k,t+l}\right]$$

and K is the number of lags we use in the cross section. Thus, we truncate the lags in the cross section, and substitute positive weights for covariances from previous periods with similar maturities.

Unfortunately, we can not apply the above formula directly, because the cross sections are of unequal sizes. Hence, the error e_{jt} need not correspond to a bond with the same maturity as the error $e_{j,t-1}$. To account for these unequal sizes, we adjust the formula above by "matching maturity." When calculating the sample covariance between e_{jt} and the errors from another date, we center the weighted summation above at the bond which most closely matches bond jt in maturity.

this procedure, and generate point estimates that are higher than those obtained proceeding period by period. In the post-1986 regime the constant tax rate should not be a significant restriction, as the U.S. tax schedule became much less progressive. Further, at the end of this section we argue that these losses in parametric freedom are not of great qualitative importance. The model we estimate allows us enough flexibility to effectively purge the data of tax effects.

4.2 Results

Table 4 Table IV: Estimates of Tax Rates and CIR parameters

parameter	estimate	standard error
ϕ_1	0.5324	.0096
ϕ_2	0.3450	.0053
ϕ_3	0.4319	.0814
tax (1978–85)	0.3086	.0041
tax (1986)	0.0010	.0061
tax (1987–92)	0.0549	.0050

The table reports estimated parameters when a separate tax rate is estimated for each of three subperiods, and the CIR parameters are held fixed over the three subperiods. Standard errors are calculated using GMM and the Newey-West weighting matrix, adjusted as described in Section II.A, with $L = 10$ and $K = 10$, where L and K are the number of lags in time-series and cross-section respectively. The number of observations is 31,258.

Table 4 IV shows estimates the CIR model for the after-tax term structure, with the structural parameters fixed through time and allowing for three different tax rates. The estimated tax rates are reported at the bottom of the table. We treat the before-tax spot rate of interest as data, as in the period-by-period estimation discussed in the previous section. The table also reports GMM standard errors for the parameters and tax rates.

The estimation over the pre-1986 subperiod produces a plausible tax rate, in contrast to the much lower average rate in the period-by-period estimation. In this sense, the multiperiod estimation identifies the tax effects as more important. In the post-1986 regime, on the contrary, the tax effects appear economically insignificant. The estimated tax rates are positive and small, while the average tax rates obtained in the period-by-period estimations are negative and close to zero. These estimates are statistically significant, based on the standard errors, but in evaluating these standard errors one should keep in mind the extremely large sample size obtained by pooling the cross sections and time series.

The estimates in Table 4 contrast with those in Table 5, which estimates the same structural model with the tax rate constrained to be the same across all periods. This produces an estimated tax rate of 22%, still higher than the average rates from the period-by-period estimation. Since the model with a single tax rate nests the model with 3 tax rates, measures of their relative fit provide a test for a regime shift associated with the 1986 Tax Reform. Under GMM assumptions, we compute a quasi-likelihood ratio statistic by taking the difference between the minimized criterion

Table 5 Estimates of Tax Rate and CIR Parameters with One Tax Rate

parameter	estimate	GMM standard error
ϕ_1	0.3405	.0037
ϕ_2	0.1540	.0017
ϕ_3	0.3732	.0075
tax (1978–92)	0.2190	.0106

The table reports parameter estimates and standard errors when the tax rate is restricted to be fixed over the whole period 1978–92. Standard errors are calculated using GMM and the Newey-West weighting matrix, adjusted as described in Section II.B, with $L = 10$ and $K = 10$, where L and K are the number of lags in time-series and cross-section respectively. The number of observations is 31,258.

functions, with and without the restriction to a single tax rate.[10]

$$LR = 2(N - H)[K(\tilde{\gamma}) - K(\hat{\gamma})] \tag{10}$$

where N is the total number of observations, H is the number of parametric restrictions, $K(\cdot)$ is the criterion function, $\tilde{\gamma}$ is the vector of six parameters, with the three tax rates constrained to be equal, and $\hat{\gamma}$ is the vector of six unrestricted parameters. In our case the LR statistic, which has an asymptotic chi-squared distribution with $N - H$ degrees of freedom, has a value of 1492.2.[11] Thus, we can conclude both that evidence for a regime shift is substantial. The differences between the tax rates estimated before and after 1986 are obviously economically significant as well.

Of course, while this procedure produces point estimates that have satisfying economic interpretations, it is possible that this comes at a considerable cost in terms of the model's ability to fit the data. Table 6 shows results from regressions of the multiperiod models' errors against characteristics of the bonds. In the top panel the dependent variable is the error from the model fit with a single tax rate, while in the bottom panel the regression is done with errors from the model fit with three tax rates.

These regressions can be compared with those in Table 3, which used the errors from fitting the model with the tax rates and structural parameters allowed to vary period by period. Estimating only three tax rates, and one set of structural parameters does produce errors are more correlated with bond characteristics than the errors from the month-by-month estimation. The increase in R^2 is only from 2.8% to 3.8%, however. Thus, the structural model with three regimes seems to do a creditable job purging the tax effects from the data. Restricting the tax rate to a single value for all periods, leaves us with an R^2 of 6.9% in this regression. In contrast, as Table 3 shows, forcing the model to fit the data with a tax rate of zero, even when the term structure parameters are free to vary through time, leaves errors for which over 36% of the variance can be explained by these

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[10]See, e.g., Davidson and MacKinnon [1993], p. 618.

[11]Under standard non-linear least squares assumptions, the ratio of the sum of squares for the restricted model (Table 5) to the unrestricted model (Table 4) has an F distribution with 2 and 31,252 degrees of freedom. The F-statistic in this case is 1474, so the inference obtained does not depend on the particular statistical assumptions employed.

characteristics of the bonds. The tax effects, accordingly, are important, but most of them are captured with a simple structural model for the after-tax term structure and three tax regimes.

Table 6 Determinants of Multiperiod Pricing Errors

	constant	coupon	maturity	age	premium	R–sqr
1 tax rate	.2489	-0.5306	0.1312	0.1587	1.7391	6.9%
	(5.35)	(-41.56)	(31.43)	(17.97)	(28.31)	
3 tax rates	.2077	-.3009	.0487	.0347	1.2603	3.8%
	(6.12)	(-32.32)	(16.01)	(5.39)	(28.13)	

The table reports the results of regressing the pricing errors from multiperiod estimations on bond characteristics associated with tax clienteles. The t-statistics are in parentheses. For each regression there are 31,258 observations.

Finally, it is quite possible that by holding fixed the non-tax parameters across subperiods, we are forcing the tax rate to reflect changes that are better attributed to instability in the structural parameters. As a check on this possibility, we estimated all the model's parameters separately across the two tax regimes. The resulting point estimates for the tax rates were 52.6% for 1978–85, -3.7% for 1986, and 2.0% for 1987–92. The fact that these tax rates show the same behaviors as those in Table IV is reassuring that the diminished tax effects subsequent to the 1986 Tax Reform Act are not a spurious consequence of other structural changes.

5   Conclusion

This paper documents the presence of tax effects in the relative pricing of U.S. Government bonds prior to 1986, and the effective disappearance of these effects after the 1986 Tax Reform. This behavior is evident in the patterns through time of point estimates of tax rates produced by fitting after-tax term structures to bond prices period by period. The quantitative significance of these effects in a statistical metric is attested by formal tests in a multi-period structural setting which uses the CIR term structure as a model of the after-tax term structure. These tests produce estimated tax rates prior to 1986 that are both statistically and economically significant, and statistical evidence of a regime shift across the pre-1986 and post-1986 periods.

There are a number of explanations that suggest themselves as to why the tax induced pricing differentials diminish so dramatically in the late 1980's. Changes in the tax regime are the most obvious, but may not be the only, factors involved. The most pronounced effect of the tax law changes in the mid-1980's was to reduce the advantages of market discounts relative to coupon income. The 1984 tax bill required investors purchasing a bond, issued after July 1984 and priced at a market discount, to amortize the discount over the life of the bond. While taxes were still only paid on sale or maturity, and could thus be deferred, capital gains and losses are thereafter determined relative to the amortized basis, rather than the total change in price. The 1986 bill then eliminated the difference in the tax rates applied to long-term capital gains and other income.

Thus, deferral of taxes was left as the only advantage to market discounts. Finally, for bonds issued after September 1985, the linear amortization of market premia was eliminated, reducing further the asymmetries between discount and premium bond. Accordingly, even if the marginal investor were taxable we would expect the differential valuation of discount, par and premium bonds to be reduced.

Simultaneously with these changes in the tax code, institutional changes were occuring that made it less likely that the marginal investor was taxable. Dealers, because their positions are marked to market and taxed symmetrically, will recognize any price differentials reflecting the taxation of individual investors as opportunities to trade profitably. The rise of derivative markets in the 1980's increased both the ability to exploit arbitrage opportunities, and the ability to hedge quasi-arbitrage positions, thus reducing their risk. Similarly, through the 1980's the number of bonds and the range of maturities issued by the U.S. Treasury increased dramatically. The higher degree of market completeness this affords would also make it easier to construct arbitrage positions and to exploit quasi-arbitrage opportunities.

The behaviors documented in this paper probably reflect both these institutional factors, which change the "marginal" investor, and revisions to the tax law, which change the relative valuations for the taxable clientele.

Appendix

Due to the statutory tax treatment of US government bonds, the formula for the model price described in the body of the paper has to be adjusted in our estimation. In this appendix we describe these adjustments. For details on these issues see Fabozzi and Nirenberg [1991].

A. Discount Bonds.

If we ignore accrued interest and amortization, the price of a discount bond is given by the present value formula

$$P_t = C(1 - \tau_i) \sum_{m=1}^{M} d_t(t_m) + 100 d_t(t_M) - (100 - P_t)\tau_g d_t(t_M), \tag{11}$$

where

$$
\begin{aligned}
P_t &= \quad \text{Bond price} \tag{12} \\
\tau_i &= \quad \text{Income tax rate} \\
\tau_g &= \quad \text{Capital gains tax rate} \\
C &= \quad \text{Coupon payment (in \$)} \\
M &= \quad \text{Number of coupon payments} \\
t_m &= \quad \text{Time of } m\text{'th coupon payment} \\
d_t(\cdot) &= \quad \text{Discount function}
\end{aligned}
$$

Solving for the price, we find

$$P_t = \frac{C(1 - \tau_i)\sum_{m=1}^{M} d_t(t_m) + 100(1 - \tau_g)d_t(t_M)}{1 - \tau_g d_t(t_M)} \tag{13}$$

We make some adjustments to this formula:

- The first coupon payment may differ from the rest. This will depend on the difference between the issue date and the maturity date for the first coupon. Let C_1 be this first coupon.

$$P_t = \frac{C_1(1 - \tau_i)d_t(t_1) + C(1 - \tau_i)\sum_{m=2}^{M} d_t(t_m) + 100(1 - \tau_g)d_t(t_M)}{1 - \tau_g d_t(t_M)} \tag{14}$$

- An adjustment for accrued interest A_t is made.

$$P_t + A_t = [(C_1 - \tau_i) + A_t\tau_i]d_t(t_1) + C(1 - \tau_i)\sum_{m=2}^{M} d_t(t_m) + 100 d_t(t_M) - (100 - P_t)\tau_g d_t(t_M). \tag{15}$$

Solving for the price we find a discount bond has a price

$$P_t^d(\tau, d_t(\cdot)) = \frac{-A_t + [C_1(1 - \tau_i) + A_t\tau_i]d_t(t_1) + C(1 - \tau_i)\sum_{m=2}^{M} d_t(t_m) + (1 - \tau_g)100d_t(t_M)}{1 - \tau_g d_t(t_M)} \quad (16)$$

For bonds issued after July 18, 1984, a market discount is amortized over the life of the bond, although both the amortized and unamortized portions of the discount are taxed upon sale or maturity. The amortized portion is taxed as income. The difference between the proceeds from sale or maturity and the amortized basis become capital gain or loss. (See page 50 of Fabozzi and Nirenberg [1991].) Accordingly, for these bonds we replace τ_g with τ_i in the above formula. This is only relevant in cases where the tax on income and capital gains differ. Note that for these cases, since we discount cash flows to maturity, the capital gains tax rate is not identifiable from bond prices since it appears in neither the theoretical discount or premium price.

$$P_t^d(\tau, d_t(\cdot)) = \frac{-A_t + [C_1(1 - \tau_i) + A_t\tau_i]d_t(t_1) + C(1 - \tau_i)\sum_{m=2}^{M} d_t(t_m) + (1 - \tau_i)100d_t(t_M)}{1 - \tau_i d_t(t_M)} \quad (17)$$

B. Premium Bonds.

The treatment of bonds selling at a premium has changed over the period in question. At issue is the timing of the recognition of the capital loss $(P_t - 100)$. If the loss was recognized at the time when the loss was realized, for a buy and hold, it would be recognized at the final maturity. In that case expression (11) would also hold for a premium bond.

The tax code does allow the holder of a bond to recognize the loss earlier. Let L_m be the loss that is recognized in period t_m. The value of a premium bond is then

$$P_t = \sum_{m=1}^{M} C(1 - \tau_i)d_t(t_m) + \sum_{m=1}^{M} L_m\tau_i d_t(t_m) + 100d_t(t_M) \quad (18)$$

Simplifying

$$P_t = \sum_{m=1}^{M} [C(1 - \tau_i) + L_m\tau_i] d_t(t_m) + 100d_t(t_M) \quad (19)$$

For bonds issued prior to September 27, 1985, the tax code prescribes *linear* amortization,

$$L_m = \frac{1}{M}(P_t - 100) \quad (20)$$

Ignoring accrued interest, solve for P_t in

$$P_t = \sum_{m=1}^{M} \left[C(1 - \tau_i) + \frac{P_t - 100}{M}\tau_i \right] d_t(t_m) + 100d_t(t_M) \quad (21)$$

23

$$P_t = \frac{\left\{C(1-\tau_i) - \frac{100}{M}\tau_i\right\}\sum_{m=1}^{M} d_t(t_m) + 100 d_t(t_m)}{(1 - \frac{\tau_i}{M})\sum_{m=1}^{M} d_t(t_m)} \tag{22}$$

The final adjustments are to account for the first coupon payment differing from the rest, and accrued interest

$$
\begin{aligned}
P_t^p(\tau, d_t(\cdot)) \;=\; & \left\{-A_t + \left[C_1(1-\tau_i) + A_t\tau_i - 100\tau_i\left(\frac{t_1 - t}{t_1 - t + \frac{1}{2}(M-1)}\right)\right] d_t(t_1)\right. \tag{23}\\
& + \left[C(1-\tau_i) - 100\left(\frac{\frac{1}{2}}{t_1 - t + \frac{1}{2}(M-1)}\right)\tau_i\right]\sum_{m=2}^{M} d_t(t_m) + \left.100 d_t(t_M)\right\} \Big/ \\
& \left\{1 - \left[\left(\frac{t_1 - t}{t_1 - t + \frac{1}{2}(M-1)}\right) d_t(t_1) + \left(\frac{\frac{1}{2}}{t_1 - t + \frac{1}{2}(M-1)}\right)\sum_{m=2}^{M} d_t(t_m)\right]\tau_i\right\}
\end{aligned}
$$

For bonds issued after September 27, 1985, the tax code requires the use of the *constant yield* method. Let B_m be the basis remaining in period m, and y the yield–to–maturity of the bond. Then

$$L_m = C - yB_m \tag{24}$$

and the basis changes according to

$$B_1 = P_t \tag{25}$$

$$B_m = B_{m-1} - (C - yB_{m-1}) = B_{m-1}(1+y) - C \tag{26}$$

(See page 52 of Fabozzi and Nirenberg [1991]). We use this formula to solve for the price P_t by first recursively defining the value L_m in terms of P_t, y and C:

$$B_m = P_t(1+y)^{m-1} - C\left\{\sum_{i=1}^{m-2}(1+y)^i\right\} \tag{27}$$

$$L_m = C - y\left(P_t(1+y)^{m-1} - C\left\{\sum_{i=1}^{m-2}(1+y)^i\right\}\right) \tag{28}$$

Substituting this L_m in (19)

$$P_t = \sum_{m=1}^{M}\left[C(1-\tau_i) + \left[C - y\left(P_t(1+y)^{m-1} - C\left\{\sum_{i=1}^{m-2}(1+y)^i\right\}\right)\right]\tau_i\right] d_t(t_m) + 100 d_t(t_M) \tag{29}$$

Now solve for P_t

$$P_t = \frac{\sum_{m=1}^{M}\left[C(1-\tau_i) + \left[C - y\left(-C\left\{\sum_{i=1}^{m-2}(1+y)^i\right\}\right)\right]\tau_i\right] d_t(t_m) + 100 d_t(t_M)}{1 + \tau_i \sum_{m=1}^{M} y(1+y)^{m-1} d_t(t_m)} \tag{30}$$

24

Simplifying,

$$P_t = \frac{\sum_{m=1}^{M} \left[C \left(1 + \tau_i y \sum_{i=1}^{m-2} (1+y)^i \right) \right] d_t(t_m) + 100 d_t(t_M)}{1 + \tau_i \sum_{m=1}^{M} y(1+y)^{m-1} d_t(t_m)} \tag{31}$$

To this formula we need to make further adjustments. To account for the possibility of a first coupon different from the rest we adjust the yield for the first period. Let

- C_1 be the first coupon, and

- y_1 be the first yield, which is assumed calculated on the basis of the time between issue and first coupon.

Ignoring accrued interest for the moment,

$$P_t = \frac{\left[C_1(1 + \tau_i y_1) \right] d_t(t_1) + \sum_{m=2}^{M} \left[C \left(1 + \tau_i y (1+y_1) \sum_{i=2}^{m-2} (1+y)^i \right) \right] d_t(t_m) + 100 d_t(t_M)}{1 + \tau_i y_1 d_t(t_1) + \tau_i y (1+y_1) \sum_{m=2}^{M} (1+y)^{m-1} d_t(t_m)} \tag{32}$$

Accrued interest simply requires that we add $-A_t + A_t \tau_i d(t_1)$ on the upper part of the above equation.

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